ON ENTROPY, REGULARITY AND RIGIDITY FOR CONVEX REPRESENTATIONS OF HYPERBOLIC MANIFOLDS

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ABSTRACT. Given a convex representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ of a convex co-compact group Γ of $\mathbb{H}^k$ we find upper bounds for the quantity αh_ρ, where h_ρ is the entropy of ρ and α is the Hölder exponent of the equivariant map $\partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d)$. We also give rigidity statements when the upper bound is attained. We then study Hitchin representations and prove that if $\rho : \pi_1 \Sigma \to \mathrm{PSL}(d, \mathbb{R})$ is in the Hitchin component then $\alpha h_\rho \leq 2/(d-1)$ (where α is the Hölder exponent of the map $\zeta : \partial_\infty \mathbb{H}^2 \to \mathscr{F}$) with equality if and only if ρ is Fuchsian.

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1. INTRODUCTION

Consider a CAT(−1) space X. Its visual boundary $\partial_\infty X$ is equipped with a natural metric, called a visual metric. This metric depends on the choice of a point in X, different points induce bi-Lipschitz equivalent metrics.

Consider now a convex co-compact action of a hyperbolic group Γ on X. An important invariant for this action is the Hausdorff dimension h_Γ for a visual metric, of the limit set L_Γ of Γ on the visual boundary $\partial_\infty X$ of X.

Several rigidity statements have been found concerning lower bounds on this Hausdorff dimension. For example, Bourdon [7] proved that if $\Gamma = \pi_1 \mathsf{M}$ where M

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is a closed k-dimensional manifold modeled on $\mathbb{H}^k$, then $h_\Gamma \geq k-1$ with equality only if the action of Γ on X preserves a totally geodesic copy of $\mathbb{H}^k$. We refer the reader to Courtois [11] for a more detailed exposition on this problem.

Given two convex-co-compact actions $\rho_i : \Gamma \to \operatorname{Isom} X_i$ $i = 1,2$ on CAT(-1) spaces X_i, there is an obvious relation between the Hausdorff dimensions of their limit sets. Let $\xi : \mathrm{L}_{\rho_1\Gamma} \to \mathrm{L}_{\rho_2\Gamma}$ be the Hölder-continuous equivariant mapping. From the definition of Hausdorff dimension one obtains

$$\alpha h_{\rho_2} \leq h_{\rho_1} \tag{1}$$

where α is the Hölder exponent of ξ, i.e. $d(\xi(x),\xi(y)) \leq K d(x,y)^\alpha$ for some $K > 0$. In this case we will say that ξ is α-Hölder.

The main purpose of this work is to extend inequality (1) for convex representations $\Gamma \to \mathrm{PGL}(d,\mathbb{R})$ and give rigidity results when the equality holds. In order to do so we will exploit the well known fact that h_Γ is also a dynamical invariant.

Consider the geodesic flow of $\Gamma\backslash X$,

$$\phi = \{\phi_t : \Gamma\backslash \mathsf{U}X \to \Gamma\backslash \mathsf{U}X\}_{t\in\mathbb{R}}.$$

The topological entropy of ϕ coincides with the Hausdorff dimension h_Γ (Sullivan [22], see also Bourdon [6]). The fact that Γ is convex-co-compact is equivalent to the fact the non wandering set of ϕ, denoted from now on $\mathsf{U}\Gamma$, is compact. Moreover, $\phi|\mathsf{U}\Gamma$ has very nice dynamical properties coming from the negative curvature of X, namely it is a topological Anosov flow (see Definition 2.4). Its topological entropy can then be computed by counting how many periodic orbits the flow $\phi|\mathsf{U}\Gamma$ has: for non torsion $\gamma \in \Gamma$ denote by

$$|\gamma| = \inf_{p\in X} d_X(p,\gamma p)$$

the length of the closed geodesic of $\Gamma\backslash X$ determined by the conjugacy class $[\gamma]$ of γ, then

$$h_\Gamma = \lim_{t\to\infty} \frac{\log \#\{[\gamma]\in[\Gamma] : |\gamma| \leq t\}}{t}.$$

Our object of study is the following:

Definition 1.1. We will say that a representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is *convex* if there exist two ρ-equivariant Hölder-continuous maps

$$\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d) \text{ and } \xi^* : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^{d^*})$$

such that $\xi(x) \oplus \ker \xi^*(y) = \mathbb{R}^d$ whenever $x, y \in \partial_\infty\Gamma$ are distinct.

We will study 2 different entropies one can define for a convex representation. For $g \in \mathrm{PGL}(d,\mathbb{R})$ denote by $\lambda_1(g)$ the logarithm of the spectral radius of g. The *spectral entropy* of a convex representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ is defined by

$$h_\rho = \lim_{t\to\infty} \frac{1}{t} \log \# \{[\gamma] \in [\Gamma] : \lambda_1(\rho\gamma) \leq t\},$$

and the *Hilbert entropy* of ρ is defined by

$$\mathsf{H}_\rho = \lim_{t\to\infty} \frac{1}{t} \log \# \left\{[\gamma]\in[\Gamma] : \frac{\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma)}{2} \leq t\right\}$$

where $\lambda_d(\rho\gamma)$ is the log of the modulus of the smallest eigenvalue of $\rho\gamma$.

One has the following Proposition:

Proposition 1.2 ([20], see also Bridgeman-Canary-Labourie-S. [10]). *The spectral entropy of an irreducible convex representation of a (finitely generated non elementary) hyperbolic group is finite and positive.*

If V is a finite dimensional vector space we will consider the distance $d_{\mathbb{P}}$ on $\mathbb{P}(V)$ coming from an euclidean distance on V. An important remark is that the entropy h_ρ of a convex representation is not necessarily the Hausdorff dimension of $\xi(\partial_\infty\Gamma)$ (see Remark 1.4 below). Our first result is the following:

Theorem A. *Let Γ be a convex co-compact group of a $\mathrm{CAT}(-1)$ space X and let $\rho:\Gamma\to\mathrm{PGL}(d,\mathbb{R})$ be an irreducible convex representation with $d\geq 3$. Then*

$$\alpha h_\rho \leq h_\Gamma \text{ and } \alpha\mathsf{H}_\rho \leq h_\Gamma,$$

when $\xi:\mathrm{L}_\Gamma\to\mathbb{P}(\mathbb{R}^d)$ is α-Hölder.

Remark that the dimension d of $\mathbb{R}^d$ does not appear in the inequality.

Consider $\mathrm{Ad}:\mathrm{PGL}(d,\mathbb{R})\to\mathrm{PGL}(\mathfrak{sl}(d,\mathbb{R}))$ the Adjoint representation. If $\rho:\Gamma\to\mathrm{PGL}(d,\mathbb{R})$ is an irreducible convex representation then $\mathrm{Ad}\,\rho:\Gamma\to\mathrm{PGL}(\mathfrak{sl}(d,\mathbb{R}))$ is not necessarily irreducible but there is a natural subspace $V_\rho\subset\mathfrak{sl}(d,\mathbb{R})$ such that

$$\mathsf{A}_\rho = \mathrm{Ad}\,\rho|V_\rho:\Gamma\to\mathrm{PGL}(V_\rho)$$

is again irreducible and convex (see Lemma 4.7). In the sequel, we will refer to A_ρ as *the irreducible adjoint representation of ρ.*

A simple computation shows that the Hilbert entropy of ρ is related to the spectral entropy of A_ρ, namely $\mathsf{H}_\rho = 2h_{\mathsf{A}_\rho}$. Nevertheless, applying this relation to the first inequality in Theorem A gives the bad upper bound $\alpha\mathsf{H}_\rho\leq 2h_\Gamma$.

Examples. There are three examples of irreducible convex representations of Γ of particular interest.

Recall that the linear subgroup $\mathrm{PSO}(1,k)$ of projective transformations preserving a bilinear form of signature $(1,k)$ is isomorphic to the isometry group $\mathrm{Isom}\,\mathbb{H}^k$ of the hyperbolic space. Throughout this work we shall refer to the representation $\overline{\phi}:\mathrm{Isom}\,\mathbb{H}^k\to\mathrm{PSO}(1,k)$ (or any of its conjugates $g\overline{\phi}g^{-1}$ with $g\in\mathrm{PGL}(k+1,\mathbb{R})$) as *the Klein model* of $\mathbb{H}^k$.

Remark 1.3. The Klein model of $\mathbb{H}^k$ induces an equivariant map $\partial_\infty\mathbb{H}^k\to\mathbb{P}(\mathbb{R}^{k+1})$. This equivariant map is a bi-Lipschitz homeomorphism onto its image.

Benoist Representations: If $\rho:\Gamma\to\mathrm{PGL}(k+1,\mathbb{R})$ preserves a proper open convex set Ω_ρ of $\mathbb{P}(\mathbb{R}^{k+1})$ and $\rho\Gamma\backslash\Omega_\rho$ is compact, then ρ is said to be a *Benoist representation*. Results from Benoist [4] imply that Benoist representations are irreducible convex representations (see [20] for details).

The Hilbert entropy of ρ is the topological entropy of the geodesic flow of $\rho\Gamma\backslash\Omega_\rho$ associated to the Hilbert metric. Crampon [12] proved that the Hilbert entropy H_ρ verifies $\mathsf{H}_\rho\leq k-1=\dim\partial\Omega_\rho$, and equality holds only when Ω_ρ is an ellipsoid, i.e. Γ acts co-compactly on $\mathbb{H}^k$ and ρ extends to the Klein model of $\mathbb{H}^k$.

Convex co-compact groups in $\mathbb{H}^k$: Consider a convex co-compact group $\phi:\Gamma\to\mathrm{Isom}\,\mathbb{H}^k$. The composition of ϕ with the Klein model of $\mathbb{H}^k$ gives rise to a convex representation $\phi':\Gamma\to\mathrm{PGL}(k+1,\mathbb{R})$.

Remark that in this setting, $\phi\Gamma$ is Zariski dense in $\mathrm{Isom}\,\mathbb{H}^k$ if and only if the action of $\phi\Gamma$ on $\mathbb{H}^k$ does not have an invariant totally geodesic copy of $\mathbb{H}^{k-1}$. If this is the case, the convex representation $\phi'\Gamma$ is irreducible.

An easy computation shows that the spectral entropy of ϕ', and the Hilbert entropy, coincide with the topological entropy of the geodesic flow of $\phi\Gamma\backslash\mathbb{H}^k$, which in turn coincides with the Hausdorff dimension of the limit set $\mathrm{L}_{\phi\Gamma}$ on $\partial_\infty\mathbb{H}^k$, (Sullivan [22]).

Assume now that $\Gamma = \pi_1\mathsf{M}$ is the fundamental group of a closed k-dimensional hyperbolic manifold, it is well known that $h_\Gamma = k-1$. Consider now a convex co-compact action $\phi : \pi_1\mathsf{M} \to \mathrm{Isom}\,\mathbb{H}^n$ with $n \geq k$ as we explained before, Bourdon states that $h_\phi \geq k-1$.

In light of the last examples one sees that a deformation of

$$\pi_1\mathsf{M} \to \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(k+1,\mathbb{R})$$

decreases Hilbert's entropy, but on the contrary, a deformation of

$$\pi_1\mathsf{M} \to \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{Isom}\,\mathbb{H}^n$$

increases Hilbert's entropy. As a conclusion the Hilbert entropy of a convex representation of $\pi_1\mathsf{M}$ may be greater or smaller than $\dim\mathsf{M}-1$, nevertheless the quantity $\alpha\mathsf{H}$ has to remain bounded by this number. Theorem A is thus optimal in this generality.

Hitchin representations and small deformations of exterior products: Consider Σ a closed oriented hyperbolic surface and say that a representation $\rho : \pi_1\Sigma \to \mathrm{PSL}(d,\mathbb{R})$ is *Fuchsian* if it factors as

$$\rho = \tau_d \circ \mathsf{f}$$

where $\tau_d : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$ the irreducible linear action (unique modulo conjugation) of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{R}^d$ and $\mathsf{f} : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R})$ is a choice of a hyperbolic metric on Σ. A *Hitchin component* of $\mathrm{PSL}(d,\mathbb{R})$ is a connected component of

$$\hom(\pi_1\Sigma, \mathrm{PSL}(d,\mathbb{R})) = \{\text{morphisms } \rho : \pi_1\Sigma \to \mathrm{PSL}(d,\mathbb{R})\}$$

containing a Fuchsian representation. As Hitchin [15] proves, representations in the Hitchin component are irreducible.

Recall that a (complete) *flag* of $\mathbb{R}^d$ is a collection of subspaces $\{V_i\}_{i=0}^d$ such that $V_i \subset V_{i+1}$ and $\dim V_i = i$. The space of flags is denoted $\mathscr{F}$. Two flags $\{V_i\}$ and $\{W_i\}$ are in *general position* if for every i one has

$$V_i \oplus W_{d-i} = \mathbb{R}^d.$$

Labourie [17] proves that if $\rho : \pi_1\Sigma \to \mathrm{PSL}(d,\mathbb{R})$ is a representation in a Hitchin component then there exists a ρ-equivariant Hölder-continuous map $\zeta : \partial_\infty\pi_1\Sigma \to \mathscr{F}$. Moreover the flags $\zeta(x)$ and $\zeta(y)$ are in general position when $x, y \in \partial_\infty\pi_1\Sigma$ are distinct.

Considering thus $\xi := \zeta_1$ the first coordinate of ζ, and $\xi^* := \zeta_d$ the last coordinate of ζ, one obtains an irreducible convex representation. Moreover, let $\Lambda^n\mathbb{R}^d$ be the n-th exterior power of $\mathbb{R}^d$. An $n-$dimensional subspace is sent to a line on $\Lambda^n\mathbb{R}^d$, hence Labourie's theorem implies that the composition $\Lambda^n\rho : \pi_1\Sigma \to \mathrm{PSL}(\Lambda^n\mathbb{R}^d)$ is again convex.

Finally, if ρ is Zariski dense on $\mathrm{PGL}(d,\mathbb{R})$, then $\Lambda^n\rho$ is irreducible. Guichard and Wienhard [14] have shown that convex irreducible representations of hyperbolic groups form an open set on the set of representations. Hence small deformations of $\Lambda^n\rho$ are still irreducible and convex.

Remark 1.4. Labourie's statement implies that if $\rho : \pi_1\Sigma \to \mathrm{PGL}(d,\mathbb{R})$ is a Hitchin representation, then the image $\xi(\partial_\infty\pi_1\Sigma)$ is a curve of class C^1 (even thought the map ξ is only Hölder). Hence, either entropy of ρ cannot be interpreted as the Hausdorff dimension of $\xi(\partial_\infty\pi_1\Sigma)$. For example, if ρ is Fuchsian, then an easy computation shows that $h_\rho = \mathsf{H}_\rho = 2/(d-1)$, even thought the limit curve is a polynomial.

1.1. **Equality.** For a convex representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ denote by

$$\alpha_\rho = \sup\{\alpha \in (0,1] : \xi : \mathrm{L}_\Gamma \to \mathbb{P}(\mathbb{R}^d) \text{ is } \alpha\text{-Hölder}\}$$

the "best" Hölder exponent of the equivariant map ξ for a visual metric on L_Γ. Remark that ξ is not necessarily α_ρ-Hölder.

A key tool to study consequences of equality in Theorem A is the *cross ratio* associated to a convex representation introduced by Labourie [18]. Given a convex representation ρ define $\mathsf{b}_\rho : (\partial_\infty\Gamma)^{(4)} \to \mathbb{R}$ by

$$\mathsf{b}_\rho(x,y,z,t) = \frac{\varphi(u)}{\psi(u)}\frac{\psi(v)}{\varphi(v)} \tag{2}$$

where $\varphi \in \xi^*(x)$, $\psi \in \xi^*(z)$, $u \in \xi(y)$ and $v \in \xi(t)$. Remark that the result does not depend on the choice of φ, ψ, u and v made.

Theorem B (Spectral entropy rigidity)**.** *Let Γ be a convex-co-compact group of $\mathbb{H}^k$ that does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$, and consider a convex irreducible representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ with $d \geq 3$ such that equality*

$$\alpha_\rho h_\rho = h_\Gamma$$

holds. Assume also that $\mathsf{b}_\rho \geq 0$. Then $d = k+1$, $\alpha_\rho = 1$ and ρ extends to $\overline{\rho} : \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(k+1,\mathbb{R})$ as the Klein model of $\mathbb{H}^k$.

Remark that if $k \geq 3$ then the condition $\mathsf{b}_\rho \geq 0$ is a necessary condition for the Theorem to hold.

A weaker statement holds for Hilbert's entropy. Recall we have defined the adjoint irreducible representation of $\mathsf{A}_\rho : \Gamma \to \mathrm{PGL}(V_\rho)$ of a given ρ, as the restriction $\mathrm{Ad}\,\rho|V_\rho$ where $V_\rho \subset \mathfrak{sl}(d,\mathbb{R})$. A slight modification of the proof of Theorem B gives the following.

Corollary 1.5 (Hilbert entropy rigidity)**.** *Let Γ be a convex-co-compact group of $\mathbb{H}^k$ that does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$, and consider a convex irreducible representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ with $d \geq 3$ such that equality*

$$\alpha_\rho \mathsf{H}_\rho = h_\Gamma$$

holds. Assume also that $\mathsf{b}_\rho \geq 0$. Then $V_\rho = \mathfrak{so}(1,k)$ and the adjoint irreducible representation $\mathsf{A}_\rho : \Gamma \to \mathrm{PGL}(\mathfrak{so}(1,k))$ extends to $\overline{\mathsf{A}_\rho} : \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(\mathfrak{so}(1,k))$ as the adjoint representation of the Klein model of $\mathbb{H}^k$.

If Γ is a co-compact group in $\mathbb{H}^2$ then the fact that the cross ratio b_ρ is non negative is consequence of the equality $\alpha_\rho h_\rho = h_\Gamma$:

Corollary 1.6. *Consider Σ a closed orientable hyperbolic surface and an irreducible convex representation $\rho : \pi_1\Sigma \to \mathrm{PGL}(d,\mathbb{R})$ with $d \geq 3$ such that $\alpha_\rho h_\rho = 1$, then $d = 3$, $\alpha_\rho = 1$ and ρ extends to the Klein model of $\mathbb{H}^2$.*

The proofs of Theorem B and Corollaries 1.5 and 1.6 are very similar and postponed to Section 11.

There are other several situations where the condition $\mathsf{b}_\rho \geq 0$ is automatically satisfied:

Corollary 1.7. *Let Γ be a convex-co-compact subgroup of $\mathbb{H}^k$ $k \geq 2$ that does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$. Denote h_Γ the Hausdorff dimension of L_Γ. Consider a convex co-compact action $\rho : \Gamma \to \operatorname{Isom} \mathbb{H}^n$ $n \geq 3$. If $\alpha_\rho h_\rho = h_\Gamma$ then ξ extends to $\overline{\xi} : \mathbb{H}^k \to \mathbb{H}^n$ as a ρ-equivariant isometric embedding.*

This is to say, $\rho\Gamma$ preserves a totally geodesic copy of $\mathbb{H}^k$ in $\mathbb{H}^n$ and moreover, the action of $\rho\Gamma$ on this geodesic copy is the departing action of Γ on $\mathbb{H}^k$.

Proof. Follows directly from Theorem B and the fact that the cross ratio $\mathbb{B}_{\mathbb{H}^n}$ of $\mathbb{H}^n$ is non negative, see Section 6.

□

Definition 1.8. Assume now that $\partial_\infty\Gamma$ is connected and that it is not the circle, we will then say that Γ *splits over a virtual* $\mathbb{Z}$ if Γ verifies either

- $\Gamma = A *_C B$ where A and B are non trivial groups and C is virtually $\mathbb{Z}$ or,
- Γ is an HNN extension over a virtual $\mathbb{Z}$.

For example, the fundamental group of a closed hyperbolic manifold of dimension ≥ 3 or a Kleinian group of $\mathbb{H}^3$ having as boundary a Sierpiński carpet, do not split over a virtual $\mathbb{Z}$ (Bowditch [8]).

Lemma 1.9 (See Lemma 6.3)**.** *Let Γ be a convex-co-compact group of $\mathbb{H}^k$ that does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$, such that $\partial_\infty\Gamma$ is connected and not homeomorphic to the circle. Assume that Γ does not split over a virtual $\mathbb{Z}$ and consider a convex irreducible representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ then $\mathsf{b}_\rho \geq 0$.*

Hence, for such groups Theorem B and Corollary 1.5 hold without the assumption $\mathsf{b}_\rho \geq 0$.

1.2. **Statements for hyperconvex representations.** The fact that equality in Corollary 1.6 can only hold for a representation $\rho : \pi_1\Sigma \to \mathrm{PSL}(3, \mathbb{R})$ suggests that the upper bound for $\alpha_\rho h_\rho$ is not optimal for Hitchin representations on $\mathrm{PSL}(d, \mathbb{R})$, say. We will now focus on improving the bound when more information on the representation ρ is given. Let G be a real algebraic semi-simple Lie group, P be a minimal parabolic subgroup of G, and write $\mathscr{F} = G/P$ for the *Furstenberg boundary* of the symmetric space of G.

Let K be a maximal compact subgroup of G, let τ be the Cartan involution on $\mathfrak{g}$ for which the set $\operatorname{fix} \tau$ is K's Lie algebra, consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$ a maximal abelian subspace contained in $\mathfrak{p}$. Let Σ be the set of (restricted) roots of $\mathfrak{a}$ on $\mathfrak{g}$. Fix $\mathfrak{a}^+$ a closed Weyl chamber and let Σ^+ a system of positive roots on Σ associated to $\mathfrak{a}^+$, denote by Π the set of simple roots associated to the choice Σ^+.

The space $\mathscr{F}$ can be embedded in a product of projective spaces $\prod_{\theta\in\Pi} \mathbb{P}(V_\theta)$ (see Section 9), we will consider the metric on $\mathscr{F}$ induced by this embedding.

The product $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit denoted by $\mathscr{F}^{(2)}$. For example, if $G = \mathrm{PGL}(d, \mathbb{R})$ then $\mathscr{F}$ is the space of complete flags of $\mathbb{R}^d$, and $\mathscr{F}^{(2)}$ is the space of flags in general position.

Definition 1.10. We say that a representation $\rho : \Gamma \to G$ is *hyperconvex* if there exists a Hölder-continuous equivariant mapping $\zeta : \partial_\infty \Gamma \to \mathscr{F}$ such that the pair $(\zeta(x), \zeta(y))$ belongs to $\mathscr{F}^{(2)}$ whenever $x, y \in \partial_\infty \Gamma$ are distinct.

Hyperconvex representations on $\mathrm{PGL}(d, \mathbb{R})$ are of course convex. As we explained before, Labourie [17] has shown that representations in a Hitchin component are hyperconvex.

The *barycenter* of the Weyl chamber $\mathfrak{a}^+$ is the half line contained in $\mathfrak{a}^+$ determined by

$$\mathsf{bar}_{\mathfrak{a}^+} = \{a \in \mathfrak{a}^+ : \theta_1(a) = \theta_2(a) \text{ for every pair } \theta_1, \theta_2 \in \Pi\}.$$

We will say that $g \in G$ is $\mathbb{R}$-*regular* if it is diagonalizable over $\mathbb{R}$, *elliptic* if it is contained in a compact subgroup of G, or *unipotent* if all its eigenvalues are equal to 1.

Recall that Jordan's decomposition states that every $g \in G$ can be written as a product $g = g_e g_h g_u$ where $g_e, g_h, g_u \in G$ commute, g_e is elliptic, g_h is $\mathbb{R}$-regular and g_u is unipotent.

For $g \in G$ denote by $\lambda(g) \in \mathfrak{a}^+$ its *Jordan projection*, this is the unique element on $\mathfrak{a}^+$ such that $\exp \lambda(g)$ is conjugated to the $\mathbb{R}$-regular element on the Jordan decomposition of g.

Again, Γ is a convex co-compact group of a $\mathrm{CAT}(-1)$ space X and h_Γ is the Hausdorff dimension of the limit set L_Γ on $\partial_\infty X$. For a hyperconvex representation $\rho : \Gamma \to G$ and a linear functional $\varphi \in \mathfrak{a}^*$ such that $\varphi|\mathfrak{a}^+ > 0$ we define the *entropy of ρ relative to φ* as

$$h_\varphi := \lim_{s \to \infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \varphi(\lambda(\rho\gamma)) \leq s\}}{s}.$$

Proposition 1.11 ([20, Section 7]). *Let $\rho : \Gamma \to G$ a Zariski dense hyperconvex representation and consider $\varphi \in \mathfrak{a}^*$ such that $\varphi|\mathfrak{a}^+ - \{0\} > 0$, then $h_\varphi \in (0, \infty)$.*

For a hyperconvex representation we have a much better bound on the quantity αh_φ.

Theorem C. *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation, and $\varphi \in \mathfrak{a}^*$ a linear functional such that $\varphi|\mathfrak{a}^+ - \{0\} > 0$ then*

$$\alpha h_\varphi \leq h_\Gamma \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\varphi(\mathsf{bar}_{\mathfrak{a}^+})}$$

where $\theta \in \Pi$ is any simple root and α is the Hölder exponent of the equivariant map $\zeta : \mathrm{L}_\Gamma \to \mathscr{F}$.

Remark that the direction of $\mathfrak{a}^+$ that gives the upper bound does not depend on the linear form φ.

In order to study rigidity when equality holds we need to restrict the functionals we consider. If χ is a restricted weight of G denote by $\Lambda_\chi : G \to \mathrm{PGL}(V_\chi)$ the irreducible proximal representation associated to it (Proposition 9.1). The composition $\Lambda_\chi \circ \rho : \Gamma \to \mathrm{PGL}(V_\chi)$ is a convex representation of Γ (Lemma 12.2), we will denote by b_χ the cross ratio defined on $\partial_\infty \Gamma$ associated to $\Lambda_\chi \circ \rho$ as in equation (2).

For a hyperconvex representation $\rho : \Gamma \to G$ denote by

$$\alpha_\rho = \sup\{\alpha \in (0, 1] : \zeta : \mathrm{L}_\Gamma \to \mathscr{F} \text{ is } \alpha\text{-Hölder}\}.$$

Theorem D. *Let Γ be a convex-co-compact group of $\mathbb{H}^k$ that does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$, and consider a Zariski dense hyperconvex representation $\rho:\Gamma\to G$. Consider a restricted weight χ of G and assume that*

$$\alpha_\rho h_\chi = h_\Gamma \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})}$$

where $\theta\in\Pi$ is any simple root. Suppose also that $\mathsf{b}_\chi\geq 0$. Then $\operatorname{Isom}\mathbb{H}^k$ is a factor of G and the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[dr] & G \arrow[d] \\ & \operatorname{Isom}\mathbb{H}^k \end{tikzcd}$$

where the arrow $\Gamma\to\operatorname{Isom}\mathbb{H}^k$ is the action of Γ on $\mathbb{H}^k$ fixed at the beginning.

As for Theorem B, the condition $\mathsf{b}_\chi\geq 0$ is not needed when Γ does not split over a virtual $\mathbb{Z}$ or when Γ act co-compactly on $\mathbb{H}^2$. A slight modification of the proof of Theorem D gives the following.

Corollary 1.12. *Let Γ be a co-compact group of $\mathbb{H}^2$ and consider a Zariski dense hyperconvex representation $\rho:\pi_1\Sigma\to G$. Suppose the equality*

$$\alpha_\rho h_\chi = \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})}$$

holds for some restricted weight χ and any simple root θ. Then $\operatorname{PSL}(2,\mathbb{R})$ is a factor of G.

Theorem C, Corollary 1.12 and a Theorem of Guichard (14.1 below) give the following corollary whose proof is postponed to the end of this article.

Corollary 1.13. *Let $\rho:\pi_1\Sigma\to\operatorname{PGL}(d,\mathbb{R})$ be a representation in the Hitchin component then*

$$\alpha_\rho h_\rho \leq \frac{2}{d-1} \text{ and } \alpha_\rho \mathsf{H}_\rho \leq \frac{2}{d-1}$$

and either equality holds only if $\rho=\tau_d\circ\mathsf{f}$, where $\tau_d:\operatorname{PSL}(2,\mathbb{R})\to\operatorname{PSL}(d,\mathbb{R})$ is the irreducible representation and $\mathsf{f}:\pi_1\Sigma\to\operatorname{PSL}(2,\mathbb{R})$ is the departing action.

Remark that if $\zeta:\partial_\infty\pi_1\Sigma\to\mathscr{F}$ is the equivariant map of a Hitchin representation then by definition, it is less (or equally) regular than the induced equivariant map $\xi=\zeta_1:\partial_\infty\pi_1\Sigma\to\mathbb{P}(\mathbb{R}^d)$. Hence, even though we obtain a much better bound on $\alpha_\rho h_\rho$ we do not know if this is produced by a decay of regularity of the map ζ.

1.3. **The method.** Let us explain the main idea for the proof of Theorem C for $G=\operatorname{PGL}(d,\mathbb{R})$ (Theorem A is proved in a similar fashion).

A Cartan subspace of $G=\operatorname{PGL}(d,\mathbb{R})$ is $\mathfrak{a}=\{(a_1,\ldots,a_d)\in\mathbb{R}^d: a_1+\cdots+a_d=0\}$, a Weyl chamber is $\mathfrak{a}^+=\{(a_1,\ldots,a_d)\in\mathfrak{a}:a_1\geq\cdots\geq a_d\}$, and the simple roots associated to $\mathfrak{a}^+$ are

$$\Pi=\{\theta_i(a_1,\ldots,a_d)=a_i-a_{i+1}: i\in\{1,\ldots,d-1\}\}.$$

The *walls* of the Weyl chamber $\mathfrak{a}^+$ are

$$\mathsf{W}_i=\{a\in\mathfrak{a}^+: a_i-a_{i+1}=0\}.$$

Denote by $\lambda : \mathrm{PGL}(d,\mathbb{R}) \to \mathfrak{a}^+$ the Jordan projection, i.e.

$$\lambda(g) = (\lambda_1(g), \dots, \lambda_d(g))$$

are the logarithm of the modulus of the eigenvalues of g, counted with multiplicity and in decreasing order.

Consider, for example, the linear functional $\varphi : \mathfrak{a}^+ \to \mathbb{R}$ defined by

$$\varphi(a_1, \dots, a_d) = \frac{a_1 - a_d}{2}.$$

Remark that h_φ, the entropy relative to φ, is the Hilbert entropy H_ρ of ρ.

For an element $g \in \mathrm{PGL}(d,\mathbb{R})$ the number

$$\frac{\theta_i(\lambda(g))}{\varphi(\lambda(g))} = 2\frac{\lambda_i(g) - \lambda_{i+1}(g)}{\lambda_1(g) - \lambda_d(g)}$$

measures, in some sense, how far the half line $\mathbb{R}_+\lambda(g)$ is from the wall W_i.

The half line which is as far as possible from *all* the walls of $\mathfrak{a}^+$ is exactly the barycenter $\mathsf{bar}_{\mathfrak{a}^+}$, this line is determined by the equalities

$$a \in \mathfrak{a}^+ : a_1 - a_2 = a_2 - a_3 = \cdots = a_{d-1} - a_d,$$

i.e. the barycenter is

$$\mathsf{bar}_{\mathfrak{a}^+} = \{((d-1)t, (d-3)t, \dots, (3-d)t, (1-d)t) : t \in \mathbb{R}_+\}.$$

For $a \in \mathsf{bar}_{\mathfrak{a}^+}$ one has

$$\frac{\theta_1(a)}{\varphi(a)} = 2\frac{a_1 - a_2}{a_1 - a_d} = \frac{2}{d-1}.$$

Hence, the upper bound on Theorem C for $\mathrm{PGL}(d,\mathbb{R})$ is $2/(d-1)$.

Given a Zariski dense hyperconvex representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ we need to find an element $\gamma \in \Gamma$ such that

$$\alpha\frac{\mathsf{H}_\rho}{h_\Gamma} \leq 2\frac{\lambda_i(\rho\gamma) - \lambda_{i+1}(\rho\gamma)}{\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma)}$$

for all $i \in \{1, \dots, d-1\}$. This is done using the Thermodynamic Formalism for convex representations developed on [20], together with some Linear Algebra of Benoist [4].

Les us explain now the main idea of Theorem B.

Consider an irreducible convex representation $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ where Γ is convex co-compact on $\mathbb{H}^k$ and does not preserve a totally geodesic copy of $\mathbb{H}^{k-1}$ (as explained before, this is equivalent to Γ being Zariski dense in $\mathrm{Isom}\,\mathbb{H}^k$). Remark we are not making any assumption on the Zariski closure of $\rho\Gamma$.

If equality $\alpha h_\rho = h_\Gamma$ holds, the Thermodynamic Formalism will imply the following relation for every non torsion $\gamma \in \Gamma$:

$$\lambda_1(\rho\gamma) = c|\gamma|$$

for some $c \in \mathbb{R}_+^*$ and where $|\gamma|$ is the translation length of $\gamma \in \Gamma$.

The modulus of the cross ratio b_ρ is uniquely determined by the spectral radii $\{\lambda_1(\rho\gamma) : \gamma \in \Gamma\}$ (see Lemma 7.3), hence the cross ratio $|\mathsf{b}_\rho|$ of the representation ρ and the cross ratio ${\mathbb{B}_{\mathbb{H}^k}}^c|\,\mathrm{L}_\Gamma$ coincide, where $\mathbb{B}_{\mathbb{H}^k}|\,\mathrm{L}_\Gamma$ is the natural cross ratio on $\partial_\infty\mathbb{H}^k$ restricted to the limit set L_Γ of Γ on $\partial_\infty\mathbb{H}^k$.

Since b_ρ is positive we have $\mathsf{b}_\rho = {\mathbb{B}_{\mathbb{H}^k}}^c|\,\mathrm{L}_\Gamma$. In oder to extend $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ to $\overline{\rho} : \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(d,\mathbb{R})$ we will use a Theorem of Labourie 6.6 which asserts that, when a cross ratio has finite rank it is induced by a convex representation.

We need then to show that the cross ratio ${\mathbb{B}_{\mathbb{H}^k}}^c$, defined on all $\partial_\infty\mathbb{H}^k$, has finite rank. In order to do so we will use some techniques from Benoist [2]. The fact that $\operatorname{Isom}\mathbb{H}^k$ is a real algebraic group is crucial in this step.

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2. Reparametrizations and Thermodynamic Formalism

Let X be a compact metric space and let $\phi = \{\phi_t : X \to X\}_{t\in\mathbb{R}}$ be a continuous flow on X without fixed points. Consider a positive continuous function $f : X \to \mathbb{R}_+^*$ and define $\kappa : X \times \mathbb{R} \to \mathbb{R}$ by

$$\kappa(x,t) = \int_0^t f\phi_s(x)ds. \tag{3}$$

The function κ has the cocycle property $\kappa(x,t+s) = \kappa(\phi_t x, s) + \kappa(x,t)$ for every $t,s \in \mathbb{R}$ and $x \in X$.

Since $f > 0$ and X is compact f has a positive minimum and $\kappa(x,\cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have an inverse $\alpha : X \times \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(x,\kappa(x,t)) = \kappa(x,\alpha(x,t)) = t \tag{4}$$

for every $(x,t) \in X \times \mathbb{R}$.

Definition 2.1. The *reparametrization* of ϕ by f is the flow $\psi = \psi^f = \{\psi_t : X \to X\}_{t\in\mathbb{R}}$ defined by $\psi_t(x) = \phi_{\alpha(x,t)}(x)$. If f is Hölder-continuous we will say that ψ is a Hölder reparametrization of ϕ.

We say that a function $U : X \to \mathbb{R}$ is C^1 *in the direction of the flow* ϕ if for every $p \in X$ the function $t \mapsto U(\phi_t(p))$ is of class C^1 and the function

$$p \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t(p))$$

is continuous. Two Hölder-continuous functions $f,g : X \to \mathbb{R}$ are then said to be *Livšic-cohomologous* if there exists a continuous $U : X \to \mathbb{R}$, of class C^1 in the direction of the flow, such that for all $p \in X$ one has

$$f(p) - g(p) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t(p)).$$

Remark 2.2. When two continuous functions $f,g : X \to \mathbb{R}_+^*$ are Livšic-cohomologous the reparametrization of ϕ by f is conjugated to the reparametrization by g, i.e. there exists a homeomorphism $h : X \to X$ such that for all $p \in X$ and $t \in \mathbb{R}$

$$h(\psi_t^f p) = \psi_t^g(hp).$$

Let ψ be the reparametrization of ϕ by $f : X \to \mathbb{R}_+^*$. If τ is a periodic orbit of ϕ then the period of τ for ψ is

$$\int_\tau f = \int_0^{p(\tau)} f(\phi_s(x))ds \tag{5}$$

where $p(\tau)$ is the period of τ for ϕ and $x \in \tau$.

If m is a ϕ-invariant probability measure on X then the probability measure $m^{\#}$ defined by

$$\frac{dm^{\#}}{dm}(\cdot) = f(\cdot) / \int f dm$$

is ψ-invariant. This relation between invariant probability measures induces a bijection and Abramov [1] relates the corresponding metric entropies:

$$h(\psi, m^{\#}) = h(\phi, m) / \int f dm. \tag{6}$$

Denote by $\mathcal{M}^{\phi}$ the set of ϕ-invariant probability measures. The *pressure* of a continuous function $f : X \to \mathbb{R}$ is defined by

$$P(\phi, f) = \sup_{m \in \mathcal{M}^{\phi}} h(\phi, m) + \int_X f dm.$$

A probability m such that the supremum is attained is called an *equilibrium state* of f. An equilibrium state for $f \equiv 0$ is called a *probability of maximal entropy*, its entropy is called the *topological entropy* of ϕ and denoted $h_{\mathrm{top}}(\phi)$.

Lemma 2.3 ([20, Section 2])**.** *Let ψ be the reparametrization of ϕ by $f : X \to \mathbb{R}_+^*$, and assume that $h_{\mathrm{top}}(\psi)$ is finite. Then $m \mapsto m^{\#}$ induces a bijection between the set of equilibrium states of $-h_{\mathrm{top}}(\psi) f$ and the set of probability measures of maximal entropy of ψ.*

2.1. **Topological Anosov flows.** For $\varepsilon > 0$ one defines the *local stable set* of x by

$$W_{\varepsilon}^{s}(x) = \{y \in X : d(\phi_t x, \phi_t y) \leq \varepsilon \ \forall t > 0 \text{ and } d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\}$$

and the local unstable set by

$$W_{\varepsilon}^{u}(x) = \{y \in X : d(\phi_{-t} x, \phi_{-t} y) \leq \varepsilon \ \forall t > 0 \text{ and } d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to \infty\}.$$

Definition 2.4. We will say that ϕ is a *topological Anosov flow* if the following holds:

- There exist positive constants C, λ and ε such that for every $x \in X$, every $y \in W_{\varepsilon}^{s}(x)$ and every $t > 0$ one has
$$d(\phi_t(x), \phi_t(y)) \leq C e^{-\lambda t}$$
and such that for every $y \in W_{\varepsilon}^{u}(x)$ one has
$$d(\phi_{-t}(x), \phi_{-t}(y)) \leq C e^{-\lambda t}.$$
- For every x there exists a neighborhood U of x such that if $y, z \in U$ then $W_{\varepsilon}^{s}(z) \cap W_{\varepsilon}^{u}(y)$ consists of at most one point and such that, for ε and $\delta > 0$ small enough, the map $W_{\varepsilon}^{s}(x) \times W_{\varepsilon}^{u}(x) \times [-\delta, \delta] \to X$ defined by
$$(y, z, t) \mapsto \phi_t(W_{\varepsilon}^{u}(y) \cap W_{\varepsilon}^{s}(z))$$
is a well defined homeomorphism onto some neighborhood of x.

A flow is said to be *transitive* if it has a dense orbit. Anosov's closing Lemma is a standard dynamical tool in hyperbolic dynamics.

Theorem 2.5 (Anosov's closing Lemma c.f. Shub [21])**.** *Let ϕ be transitive topological Anosov flow, then periodic orbits are dense in the set of ergodic invariant probability measures of ϕ.*

The following is standard in the study of Ergodic Theory of Anosov flows.

Proposition 2.6 (Bowen-Ruelle [9]). *Let ϕ be a transitive topological Anosov flow. Then given a Hölder-continuous function $f : X \to \mathbb{R}$ there exists a unique equilibrium state for f, moreover, the equilibrium state is ergodic. If two functions have the same equilibrium state then their difference is Livšic-cohomologous to a constant.*

We will need the following immediate Lemma.

Lemma 2.7. *Let ϕ be a topological Anosov flow on X and $f : X \to \mathbb{R}_+$ a positive Hölder-continuous function. Denote by*

$$h_f = \lim_{t\to\infty} \frac{1}{t} \log \#\{\tau\text{-periodic} : \int_\tau f \leq t\},$$

then

$$\frac{h(\phi, m_{-h_f f})}{h_f} = \int f dm_{-h_f f}.$$

Proof. Let ϕ^f be the reparametrization of ϕ by f. The flow ϕ^f is still a topological Anosov flow and hence its topological entropy can then be computed as the exponential growth rate of its periodic orbits, i.e. the topological entropy of ϕ^f is h_f (recall equation (5)). The Lemma finishes applying Lemma 2.3 and Abramov's formula (6). □

3. CAT(−1) SPACES

The standard reference for this section is Bourdon [6]. Consider a CAT(−1) space X and $\partial_\infty X$ its visual boundary. The *Busseman function* of X, $B : \partial_\infty X \times X \times X \to \mathbb{R}$ is defined by

$$B(z, p, q) = B_z(p, q) = \lim_{s\to\infty} d_X(p, \sigma(s)) - d_X(q, \sigma(s))$$

where $\sigma : [0, \infty) \to X$ is any geodesic ray such that $\sigma(\infty) = z$.

Denote by

$$\partial^{(2)}_\infty X = \partial_\infty X \times \partial_\infty X - \{(x, x) : x \in \partial_\infty X\}$$

and fix a point $o \in X$. The *Gromov product* of X based on o, $[\cdot, \cdot]_o : \partial^{(2)}_\infty X \to \mathbb{R}$, is defined by

$$[x, y]_o = \frac{1}{2}(B_x(o, p) + B_y(o, p))$$

where p is any point in the geodesic joining x and y. Remark that $[x, y]_o \to \infty$ as y approaches x. The *visual metric* based on o on $\partial_\infty X$ is defined by $\delta_o(x, y) = e^{-[x,y]_o}$. Since X is CAT(−1) this is in fact a distance on $\partial_\infty X$.

For $\gamma \in \operatorname{Isom} X$ denote by $|\gamma|$ its translation length

$$|\gamma| = \inf_{p\in X} d_X(p, \gamma p).$$

If γ is hyperbolic then one has $|\gamma| = B_{\gamma_+}(\gamma^{-1}o, o)$ for any $o \in X$ where γ_+ is the attractor of γ on $\partial_\infty X$.

The following Lemma shows that the translation length of a hyperbolic element can be directly computed from its action on the visual boundary of X.

Lemma 3.1. *Consider a hyperbolic element $\gamma \in \operatorname{Isom} X$ then for any $x \in \partial_\infty X - \{\gamma_-\}$ one has*

$$\lim_{n\to\infty} \frac{\log \delta_o(\gamma^n x, \gamma_+)}{n} = -|\gamma|.$$

Proof. This is standard (Yue [24]). Fix two points $x, z \in \partial_\infty X$, then for every $\gamma \in \operatorname{Isom} X$ one has

$$\delta_o(\gamma z, \gamma x) = e^{\frac{1}{2}(B_{\gamma z}(\gamma o, o) + B_{\gamma x}(\gamma o, o))}\delta_o(z, x).$$

Hence, for each δ there exists a neighborhood V of z such that for every $x \in V$ one has

$$1 - \delta \leq \frac{\delta_0(\gamma z, \gamma x)}{\delta_o(z, x)} e^{-B_{\gamma z}(\gamma o, o)} \leq 1 + \delta.$$

Assume now that γ is a hyperbolic element and that $z = \gamma_+$. Fix δ and assume that $x \in V$, then one has

$$(1-\delta)^n \leq \frac{\delta_o(\gamma_+, \gamma^n x)}{\delta_o(\gamma_+, x)} e^{-nB_{\gamma_+}(\gamma o, o)} \leq (1+\delta)^n.$$

Taking logarithm and dividing by n one obtains the desired conclusion. If $x \notin V$ then a big enough power $\gamma^N x$ does (recall $x \neq \gamma_-$) and one repeats the argument. □

For a discrete subgroup Γ of $\operatorname{Isom} X$ denote by L_Γ its limit set on $\partial_\infty X$. Consider the space $\widetilde{\mathsf{U}\Gamma}$ defined by

$$\{\sigma : (-\infty, \infty) \to X : \sigma \text{ is a complete geodesic with } \sigma(-\infty), \sigma(\infty) \in \mathrm{L}_\Gamma\}.$$

The group Γ naturally acts on $\widetilde{\mathsf{U}\Gamma}$ and we denote by $\mathsf{U}\Gamma = \Gamma \backslash \widetilde{\mathsf{U}\Gamma}$ its quotient. We will say that Γ is *convex co-compact* if the space $\mathsf{U}\Gamma$ is compact.

Remark 3.2. Throughout this work we will fix a convex co-compact action of Γ on X, hence we allow ourselves to naturally identify L_Γ to $\partial_\infty \Gamma$ and to refer to the space $\mathsf{U}\Gamma$ as only depending on Γ.

The space $\mathsf{U}\Gamma$ is naturally equipped with a flow $\phi = \{\phi_t : \mathsf{U}\Gamma \to \mathsf{U}\Gamma\}_{t \in \mathbb{R}}$ simply by changing the parametrization of a given complete geodesic. This is called *the geodesic flow of* Γ

The following Theorem relates this section to the preceding one:

Theorem 3.3 (c.f. Bourdon [6])**.** *Let Γ be a convex-co-compact group of X. Then the geodesic flow of Γ is a topological Anosov flow. The topological entropy of the geodesic flow is hence*

$$h_\Gamma = \lim_{s \to \infty} \frac{\log \#\{[\gamma] \in [\Gamma] \text{ non torsion} : |\gamma| \leq s\}}{s}.$$

3.1. **Hölder cocycles.** We shall now study Hölder cocycles on $\partial_\infty \Gamma$. The main references for this subsection are Ledrappier [19] and [20, Section 5].

Definition 3.4. A *Hölder cocycle* is a function $c : \Gamma \times \partial_\infty \Gamma \to \mathbb{R}$ such that

$$c(\gamma_0 \gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x)$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial_\infty \Gamma$, and where $c(\gamma, \cdot)$ is a Hölder map for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Given a Hölder cocycle c and a non torsion element $\gamma \in \Gamma$ we define the *period* of γ for c by

$$\ell_c(\gamma) = c(\gamma, \gamma_+)$$

where γ_+ is the attractive fixed point of γ on $\partial_\infty \Gamma$. The cocycle property implies that the period of an element γ only depends on its conjugacy class $[\gamma] \in [\Gamma]$.

We shall be interested in cocycles whose periods are positive, i.e. such that $\ell_c(\gamma) > 0$ for every non torsion $\gamma \in \Gamma$. The *entropy*† of such cocycle is defined by:

$$h_c = \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] : \ell_c(\gamma) \leq t\} \in \mathbb{R}_+ \cup \{\infty\}.$$

Theorem 3.5 ([20, Section 3])**.** *Let c be a Hölder cocycle such that $h_c \in (0, \infty)$, then there exists a positive Hölder-continuous function $f : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ such that*

$$\int_{[\gamma]} f = \ell_c(\gamma)$$

for every non torsion $[\gamma]$.

4. Convex representations

Let Γ be a convex co-compact group of a CAT(-1) space.

Definition 4.1. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *convex* if there exists a ρ-equivariant Hölder-continuous map

$$(\xi, \xi^*) : \partial_\infty\Gamma \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*}),$$

such that $\mathbb{R}^d = \xi(x) \oplus \ker \xi^*(y)$ whenever $x \neq y$.

Lemma 4.2. *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a convex representation, then the action of $\rho\Gamma$ on $\langle \xi(\partial_\infty\Gamma) \rangle$ is irreducible.*

Proof. Consider $W \subset \langle \xi(\partial_\infty\Gamma) \rangle$ a $\rho\Gamma$-invariant subspace. Consider $w \in W$ and write

$$w = \sum_{i=1}^{k} \alpha_i v_i$$

where $v_i \in \xi(x_i)$ for k-points $x_i \in \partial_\infty\Gamma$. Consider now some non torsion $\gamma \in \Gamma$ such that $\gamma_- \notin \{x_1, \ldots, x_k\}$. We then have $\gamma^n x_i \to \gamma_+$ and hence $\mathbb{R}\rho\gamma^n(w) \to \xi(\gamma_+)$ in $\mathbb{P}(\mathbb{R}^d)$. Thus $\xi(\gamma_+) \in W$, since W is $\rho\Gamma$-invariant one has

$$\xi(\partial_\infty\Gamma) = \xi(\overline{\Gamma \cdot \gamma_+}) \subset W.$$

This finishes the proof.

□

We say that $g \in \mathrm{PGL}(d, \mathbb{R})$ is *proximal* if it has a unique complex eigenvalue of maximal modulus, and its generalized eigenspace is one dimensional. This eigenvalue is necessarily real and its modulus is equal to $\exp \lambda_1(g)$. We will denote by g_+ the g-fixed line of $\mathbb{R}^d$ consisting of eigenvectors of this eigenvalue, and g_- the g-invariant complement of g_+ (i.e. $\mathbb{R}^d = g_+ \oplus g_-$). The line g_+ is an attractor on $\mathbb{P}(\mathbb{R}^d)$ for the action of g and g_- is a repelling hyperplane.

Lemma 4.3 ([20, Section 3])**.** *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a convex irreducible representation. Then for every non torsion element $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal, $\xi(\gamma_+)$ is its attractive fixed line and $\xi^*(\gamma_-)$ is the repelling hyperplane. Consequently $\xi(x) \subset \xi^*(x)$ for every $x \in \partial_\infty\Gamma$.*

†In [20] this is called the exponential growth rate of the cocycle.

Fix now some norm $\| \ \|$ on $\mathbb{R}^d$. We define the Hölder cocycles $\beta_\rho, \overline{\beta}_\rho : \Gamma \times \partial_\infty \Gamma \to \mathbb{R}$ by

$$\beta_\rho(\gamma, x) = \log \frac{\|\rho(\gamma)v\|}{\|v\|} \text{ and } \overline{\beta}_\rho(\gamma, x) = \log \frac{\|\theta \circ \rho(\gamma^{-1})\|}{\|\theta\|}$$

for some non zero $v \in \xi(x)$, and a non zero functional $\theta \in \xi^*(x)$. Lemma 4.3 implies the following.

Lemma 4.4 ([20, Section 3])**.** *Assume ρ is convex and irreducible, then the period of a non torsion $\gamma \in \Gamma$ for β_ρ is $\beta_\rho(\gamma, \gamma_+) = \lambda_1(\rho\gamma)$, and the period $\overline{\beta}_\rho(\gamma, \gamma_+) = \lambda_1(\rho\gamma^{-1}) = -\lambda_d(\rho\gamma)$.*

Proposition 1.2 together with the last Lemma and Theorem 3.5 imply the following.

Corollary 4.5 ([20, Section 5])**.** *Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be an irreducible convex representation, then there exists a positive Hölder-continuous function $f : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ such that*

$$\int_{[\gamma]} f = \lambda_1(\rho\gamma)$$

for every non torsion $[\gamma]$.

4.1. **Adjoint representation.** Given an irreducible convex representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ we will now show how the Adjoint representation $\mathrm{Ad} : \mathrm{PGL}(d, \mathbb{R}) \to \mathrm{PGL}(\mathfrak{sl}(d, \mathbb{R}))$ induces again an irreducible convex representation A_ρ such that

$$\lambda_1(\mathsf{A}_\rho\gamma) = \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma).$$

This is standard.

Recall that the *adjoint representation* is defined by conjugation $\mathrm{Ad}(g)(T) = gTg^{-1}$, where $T \in \mathfrak{sl}(d, \mathbb{R}) = \{\text{traceless endomorphisms of } \mathbb{R}^d\}$. Consider $\mathscr{F}_*(\mathbb{R}^d)$ the space of incomplete flags consisting of a line contained on a hyperplane,

$$\mathscr{F}_*(\mathbb{R}^d) = \{(v, \theta) \in \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*}) : \theta(v) = 0\}.$$

Given $(v, \theta) \in \mathscr{F}_*$ define $M(v, \theta) \in \mathbb{P}(\mathfrak{sl}(d, \mathbb{R}))$ by $M(v, \theta)(w) = \theta(w)v$ and define $\Phi(v, \theta) \in \mathbb{P}(\mathfrak{sl}(d, \mathbb{R})^*)$ by $\Phi(v, \theta)(T) = \theta(Tv)$. These maps induce a map

$$(M, \Phi) : \mathscr{F}_*(\mathbb{R}^d) \to \mathscr{F}_*(\mathfrak{sl}(d, \mathbb{R})).$$

Say that two points $(v, \theta), (w, \varphi) \in \mathscr{F}_*(\mathbb{R}^d)$ are *in general position* if

$$\theta(w) \neq 0 \text{ and } \varphi(v) \neq 0.$$

Lemma 4.6. *The maps M and Φ are* Ad*-equivariant. If $(v, \theta), (w, \varphi) \in \mathscr{F}_*(\mathbb{R}^d)$ are in general position then the points*

$$(M, \Phi)(v, \theta) \text{ and } (M, \Phi)(w, \varphi)$$

are also in general position. If g and g^{-1} are proximal then $\mathrm{Ad}\, g$ is proximal and its attractor $(\mathrm{Ad}\, g)_+$ is $M(g_+, (g^{-1})_-)$.

The proof of the Lemma is standard and direct.

Lemma 4.7. *Consider a convex irreducible representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ and consider the map $\eta = M \circ (\xi, \xi^*) : \partial_\infty \Gamma \to \mathbb{P}(\mathfrak{sl}(d, \mathbb{R}))$. Denote by $V = \langle \eta(\partial_\infty \Gamma) \rangle$ and*

$$\eta^* = (\Phi \circ (\xi, \xi^*)) \cap V.$$

Then $\mathsf{A}_\rho = \mathrm{Ad}\circ\rho|V : \Gamma \to \mathrm{PGL}(V)$ *is an irreducible convex representation with equivariant maps* (η,η^*)*, moreover for a non torsion* $\gamma\in\Gamma$ *then*

$$\lambda_1(\mathsf{A}_\rho\gamma) = \lambda_1(\rho\gamma) - \lambda_d(\rho\gamma).$$

We will say that A_ρ is the *irreducible adjoint representation of* ρ.

Proof. Irreducibility follows from Lemma 4.2 The other properties are consequence of Lemma 4.6 together with Lemma 4.3. The last statement follows from the fact that if $\gamma\in\Gamma$ is non torsion then $\xi^*(\gamma_+)$ is the repelling hyperplane of $\rho\gamma^{-1}$ and the attractor of A_ρ,

$$M(\xi(\gamma_+),\xi^*(\gamma_+)),$$

belongs to V. □

Remark 4.8. Remark that the Hilbert entropy verifies $\mathsf{H}_\rho = 2h_{\mathsf{A}_\rho} < \infty$.

Hence, using Theorem 4.5 one obtains:

Corollary 4.9. *Let* $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ *be an irreducible convex representation, then there exists a positive Hölder-continuous function* $f : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ *such that*

$$\int_{[\gamma]} f = \frac{\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma)}{2}$$

for every non torsion $[\gamma]$.

4.2. **Regularity.** The following Lemma is from Benoist [4].

Lemma 4.10 (Benoist [4])**.** *Let* $g\in\mathrm{PGL}(V)$ *be proximal and let* $V_{\lambda_2(g)}$ *be the sum of the characteristic spaces of* g *whose eigenvalue is of modulus* $\exp\lambda_2(g)$*. Then for every* $v\notin\mathbb{P}(g_-)$ *with non zero component in* $V_{\lambda_2(g)}$ *one has*

$$\lim_{n\to\infty}\frac{\log d_{\mathbb{P}}(g^n(v),g_+)}{n} = \lambda_2(g)-\lambda_1(g).$$

The following Lemma is the key to relate the Hölder exponent of the equivariant map of a convex representation and eigenvalues of $\rho(\gamma)$ for non torsion $\gamma\in\Gamma$.

Lemma 4.11. *Let* $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ *be a convex irreducible representation and denote by* α *the Hölder exponent of the equivariant mapping* ξ*, then for every no torsion* $\gamma\in\Gamma$ *one has*

$$\alpha \leq \min\left\{\frac{\lambda_1(\rho\gamma)-\lambda_2(\rho\gamma)}{|\gamma|}, \frac{\lambda_{d-1}(\rho\gamma)-\lambda_d(\rho\gamma)}{|\gamma|}\right\}.$$

Proof. Consider a non torsion $\gamma\in\Gamma$. Since ρ is irreducible there exists $x\in\partial_\infty\Gamma - \{\gamma_-\}$ such that $\xi(x)$ has non zero projection to $V_{\lambda_2(\rho\gamma)}$, the characteristic space of $\rho\gamma$ of eigenvalue of modulus $\exp\lambda_2(\rho\gamma)$. Lemma 4.3 states that $\xi(\gamma_+)$ is the attractor of $\rho\gamma$, hence applying Benoist's Lemma 4.10 we obtain

$$\lambda_2(\rho\gamma)-\lambda_1(\rho\gamma) = \lim_{n\to\infty}\frac{\log d_{\mathbb{P}}(\rho\gamma^n(\xi x),\xi(\gamma_+))}{n} = \lim_{n\to\infty}\frac{\log d_{\mathbb{P}}(\xi(\gamma^n x),\xi(\gamma_+))}{n}$$

since ξ is equivariant. Hölder continuity of ξ implies that the last quantity is smaller than

$$\lim_{n\to\infty}\frac{\log K\delta_o(\gamma^n x,\gamma_+)^\alpha}{n} = -\alpha|\gamma|$$

according to Lemma 3.1. Thus, for every $\gamma\in\Gamma$, one has

$$\alpha\leq\frac{\lambda_1(\rho\gamma)-\lambda_2(\rho\gamma)}{|\gamma|},$$

applying this inequality to γ^{-1} one obtains

$$\alpha\leq\frac{\lambda_{d-1}(\rho\gamma)-\lambda_d(\rho\gamma)}{|\gamma|}.$$

□

5. Proof of Theorem A

This section is devoted to the proof of Theorem A. Consider an irreducible convex representation $\rho:\Gamma\to\mathrm{PGL}(d,\mathbb{R})$ and denote by c either the Hölder cocycle

$$\beta_\rho \text{ or } \frac{\beta_\rho+\overline{\beta}_\rho}{2},$$

and denote by $\ell_c(\gamma)$ the period of a non torsion γ for c. This is to say, either $\ell_c(\gamma)$ equals $\lambda_1(\rho\gamma)$ or either it equals $(\lambda_1(\rho\gamma)-\lambda_d(\rho\gamma))/2$.

Recall that

$$h_c=\lim_{t\to\infty}\frac{\log\#\{[\gamma]\in[\Gamma] \text{ non torsion} : \ell_c(\gamma)\leq t\}}{t}.$$

According to Corollary 4.5 or to Corollary 4.9 (depending on whether $c=\beta_\rho$ or $c=(\beta_\rho+\overline{\beta}_\rho)/2$) there exists a positive Hölder-continuous function $f:\mathsf{U}\Gamma\to\mathbb{R}_+^*$ such that for every non torsion conjugacy class $[\gamma]$ of $[\Gamma]$ one has

$$\int_{[\gamma]}f=\ell_c(\gamma).$$

Let m_{-h_cf} denote the equilibrium state of $-h_cf$ and consider a sequence of periodic orbits $\{[\gamma_n]\}$ such that

$$\frac{\mathrm{Leb}_{\gamma_n}}{|\gamma_n|}\to m_{-h_cf}$$

as $n\to\infty$. The existence of this sequence is guaranteed by ergodicity of equilibrium states (Proposition 2.6) and Anosov's closing Lemma 2.5.

One thus obtains

$$\frac{\ell_c(\gamma_n)}{|\gamma_n|}=\frac{1}{|\gamma_n|}\int_{[\gamma_n]}f\to\int f dm_{-h_cf}$$

which, using Lemma 2.7, is equal to

$$\frac{h(\phi,m_{-h_cf})}{h_c}.$$

Hence, given $\varepsilon>0$ one has

$$\frac{\ell_c(\gamma_n)}{|\gamma_n|}\leq\frac{h(\phi_t,m_{-h_cf})}{h_c}(1+\varepsilon)$$

for all n large enough. Using Lemma 4.11 one has

$$\alpha\leq\min\left\{\frac{(\lambda_1-\lambda_2)(\rho\gamma_n)}{|\gamma_n|},\frac{(\lambda_{d-1}-\lambda_d)(\rho\gamma_n)}{|\gamma_n|}\right\}$$
$$\leq\min\left\{\frac{(\lambda_1-\lambda_2)(\rho\gamma_n)}{\ell_c(\gamma)},\frac{(\lambda_{d-1}-\lambda_d)(\rho\gamma_n)}{\ell_c(\gamma)}\right\}\frac{h(\phi,m_{-h_cf})}{h_c}(1+\varepsilon).$$

Theorem 3.3 states that h_Γ is the maximal entropy of the flow ϕ on $\mathsf{U}\Gamma$, hence

$$\frac{\alpha h_c}{h_\Gamma} \leq \frac{\alpha h_c}{h(\phi, m_{-h_c f})} \leq \min\left\{\frac{(\lambda_1-\lambda_2)(\rho\gamma_n)}{\ell_c(\gamma)}, \frac{(\lambda_{d-1}-\lambda_d)(\rho\gamma_n)}{\ell_c(\gamma)}\right\}(1+\varepsilon) \tag{7}$$

for all n large enough.

We will now distinguish the two cases:

First case: $c=\beta_\rho$**.** In this case $\ell_c(\gamma)=\lambda_1(\rho\gamma)$, $h_c=h_\rho$ (the spectral entropy of ρ) and equation (7) is

$$\frac{\alpha h_\rho}{h_\Gamma} \leq \frac{\alpha h_\rho}{h(\phi, m_{-h_\rho f})} \leq \min\left\{\frac{\lambda_1-\lambda_2}{\lambda_1}(\rho\gamma_n), \frac{\lambda_{d-1}-\lambda_d}{\lambda_1}(\rho\gamma_n)\right\}(1+\varepsilon).$$

We will now maximize the function $\mathsf{V}_1:\mathbb{P}(\mathfrak{a}^+)\to\mathbb{R}$ defined by

$$\mathsf{V}_1(a_1,\dots,a_d)=\min\left\{\frac{a_1-a_2}{a_1}, \frac{a_{d-1}-a_d}{a_1}\right\}.$$

Recall that

$$\mathfrak{a}^+=\{(a_1,\dots,a_d)\in\mathbb{R}^d : a_1+\cdots+a_d=0 \text{ and } a_1\geq\cdots\geq a_d\}$$

and consider $a\in\mathfrak{a}^+$. We will distinguish two cases.

Assume $a_2\geq 0$: In this case one has

$$\mathsf{V}_1(a)\leq\frac{a_1-a_2}{a_1}=1-\frac{a_2}{a_1}\leq 1.$$

Assume $a_2<0$:

Remark 5.1. In this case one has $a_1-a_2>a_{d-1}-a_d$, hence $\mathsf{V}_1(a)=(a_{d-1}-a_d)/a_1$.

Proof. Remark that $a_{k+1}-a_k\leq 0$ for all $k\in\{1,\dots,d-1\}$. Using the following tricky equality (recall $d\geq 3$)

$$a_1+(d-1)a_2+\sum_{k=2}^{d-1}(d-k)(a_{k+1}-a_k)=a_1+a_2+\cdots+a_d=0$$

one obtains

$$a_1-a_2+a_d-a_{d-1}=-da_2-\sum_{k=2}^{d-2}(d-k)(a_{k+1}-a_k)>0.$$

Hence $a_1-a_2>a_{d-1}-a_d$.

□

Since $0>a_2\geq\cdots\geq a_d$ one has

$$a_1=-a_2-\cdots-a_d>-(a_{d-1}+a_d)\geq 0.$$

Given that $d\geq 3$ one obtains, $a_{d-1}<0<-a_{d-1}$ and subtracting a_d on each side one gets $a_{d-1}-a_d<-(a_{d-1}+a_d)<a_1$, finally

$$\mathsf{V}_1(a)=\frac{a_{d-1}-a_d}{a_1}<1.$$

In any case one obtains $\mathsf{V}_1\leq 1$. We then get

$$\frac{\alpha h_\rho}{h_\Gamma}\leq\frac{\alpha h_\rho}{h(\phi, m_{-h_\rho f})}\leq\mathsf{V}_1(\lambda(\rho\gamma_n))(1+\varepsilon)\leq 1+\varepsilon, \tag{8}$$

since ε is arbitrary we obtain the desired inequality.

Remark 5.2. Consider

$$\alpha_\rho = \sup\{\alpha \in (0,1] : \xi \text{ is } \alpha\text{-Hölder}\}.$$

We remark that if $\alpha_\rho h_\rho = h_\Gamma$ then inequality (8) implies that $m_{-h_\rho f}$ is the measure of maximal entropy of ϕ and thus (using Proposition 2.6) the function $f : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ is Livšic-cohomologous to a constant, i.e. there exists $c > 0$ such that

$$\lambda_1(\rho\gamma) = c|\gamma|$$

for every non torsion $\gamma \in \Gamma$.

Second case: $c = (\beta_\rho + \overline{\beta}_\rho)/2$**.** In this case we have $\ell_c(\gamma) = (\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma))/2$, $h_c = \mathsf{H}_\rho$ (the Hilbert entropy of ρ) and inequality (7) states

$$\frac{\alpha \mathsf{H}_\rho}{h_\Gamma} \leq \frac{\alpha \mathsf{H}_\rho}{h(\phi, m_{-\mathsf{H}_\rho f})} \leq \min\left\{\frac{\lambda_1 - \lambda_2}{(\lambda_1 - \lambda_d)/2}(\rho\gamma_n), \frac{\lambda_{d-1} - \lambda_d}{(\lambda_1 - \lambda_d)/2}(\rho\gamma_n)\right\}(1+\varepsilon)$$

for all n large enough.

We will now maximize the function $\mathsf{V}_2 : \mathbb{P}(\mathfrak{a}^+) \to \mathbb{R}$ defined by

$$\mathsf{V}_2(a_1, \ldots, a_d) = \min\left\{\frac{a_1 - a_2}{(a_1 - a_d)/2}, \frac{a_{d-1} - a_d}{(a_1 - a_d)/2}\right\}.$$

Consider $a \in \mathfrak{a}^+$ such that

$$x := a_1 - a_2 \leq a_{d-1} - a_d =: y.$$

For such a one has $a_2 = a_1 - x$ and $a_{d-1} = y + a_d$. Since $d \geq 3$ one has $a_2 \geq a_{d-1}$ hence $a_1 - x \geq a_d + y \geq a_d + x$ and thus

$$\mathsf{V}_2(a) = \frac{2x}{a_1 - a_d} \leq 1.$$

If, on the opposite, one has $a \in \mathfrak{a}^+$ such that

$$x := a_{d-1} - a_d \leq a_1 - a_2 =: y,$$

then, again the fact that $a_2 \geq a_{d-1}$ implies $a_1 - x \geq a_1 - y \geq a_d + x$ and thus

$$\mathsf{V}_2(a) = \frac{2x}{a_1 - a_d} \leq 1.$$

In any case one obtains $\mathsf{V}_2 \leq 1$. We then get

$$\frac{\alpha \mathsf{H}_\rho}{h_\Gamma} \leq \frac{\alpha \mathsf{H}_\rho}{h(\phi, m_{-\mathsf{H}_\rho f})} \leq \mathsf{V}_2(\lambda(\rho\gamma_n))(1+\varepsilon) \leq 1 + \varepsilon, \tag{9}$$

since ε is arbitrary we obtain the desired inequality.

Remark 5.3. Remark that if $\alpha_\rho \mathsf{H}_\rho = h_\Gamma$ then inequality (9) implies that $m_{-\mathsf{H}_\rho f}$ is the measure of maximal entropy of ϕ and thus (using Proposition 2.6) the function $f : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ is Livšic-cohomologous to a constant, i.e. there exists $c > 0$ such that

$$\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) = 2c|\gamma|$$

for every non torsion $\gamma \in \Gamma$.

6. Cross ratios: General Lemmas and a Theorem of Labourie

Consider a compact metric space X and

$$X^{(4)} = \{(x,y,z,t) \in X^4 \text{ such that } x \neq t \text{ and } y \neq z\}$$

Definition 6.1. A cross ratio on X is a Hölder-continuous function $\mathsf{b} : X^{(4)} \to \mathbb{R}$ that verifies the following relations for every $x,y,z,t,w \in X$:

- $\mathsf{b}(x,y,z,t) = \mathsf{b}(z,t,x,y)$,
- $\mathsf{b}(x,y,x,t) = 1 = \mathsf{b}(x,y,z,y)$,†
- $\mathsf{b}(x,y,z,t) = 0$ if and only if $x = y$ or $z = t$,
- $\mathsf{b}(x,y,z,t) = \mathsf{b}(x,y,z,w)\mathsf{b}(x,w,z,t)$,
- $\mathsf{b}(x,y,z,t) = \mathsf{b}(x,y,w,t)\mathsf{b}(w,y,z,t)$.

Remark that if b is a cross ratio then so are b^n for $n \in \mathbb{Z}$, its modulus $|\mathsf{b}|$, and $|\mathsf{b}|^c$ for any positive real number c.

The following Lemma is trivial.

Lemma 6.2. *Consider a cross ratio b on X and assume a connected group G acts on X, then for every $g \in G$ and $(x,y,z,t) \in X^{(4)}$ the sign of $\mathsf{b}(x,y,z,t)$ and the sign of $\mathsf{b}(gx,gy,gz,gt)$ coincide.*

Remark that if for every pair of points $x,y \in X$ the space $X - \{x,y\}$ is connected then any cross ratio b defined on X is necessarily non negative. Indeed, fixing $x,y,z \in X$ pairwise distinct and considering $\mathsf{b}(x,\cdot,y,z) : X - \{y\} \to \mathbb{R}$ one obtains a function that only vanishes on x, i.e. $\mathsf{b}(x,\cdot,y,z) : X - \{x,y\} \to \mathbb{R}$ is continuous, non vanishing and takes the value 1 on z. Analog reasoning works for the other entries of b.

Lemma 6.3. *Let Γ be a hyperbolic group that does not split over a cyclic $\mathbb{Z}$ (recall definition 1.8), then any cross ratio defined on $\partial_\infty\Gamma$ is non negative.*

Proof. This is direct consequence of the preceding paragraph and following deep fact of Bowditch [8]: the boundary of such group does not have local cut points on $\partial_\infty\Gamma$, which in turn implies that $\partial_\infty\Gamma - \{x,y\}$ is connected for any pair of points on $x,y \in \partial_\infty\Gamma$. □

The rank of a cross ratio is defined as follows. Consider X^{p*} the set of pairs $(e,u) = (e_0,\dots,e_p,u_0,\dots,u_p)$ of $p+1$-tuples on X such that $e_j \neq e_i \neq u_0$ and $u_j \neq u_i \neq e_0$ when $j > i > 0$, define then

$$\chi^p_{\mathsf{b}}(e,u) = \det_{i,j>0}(\mathsf{b}(e_i,u_j,e_0,u_0)).$$

Definition 6.4. The *rank*‡ of a cross ratio b on X is defined by

$$\operatorname{rank}\mathsf{b} = \inf\{p \in \mathbb{N} : \chi^p_{\mathsf{b}} \equiv 0\} - 1,$$

if it exists.

Consider $d \in \mathbb{N}$, a *convex map* from X to $\mathbb{P}(\mathbb{R}^d)$ is a Hölder-continuous map

$$(\xi,\xi^*) : X \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*})$$

such that $\xi(x) \subset \ker\xi^*(y)$ if and only if $x = y$.

† This condition is weaker than the analog one defined on Labourie [17].
‡ This is a slightly different definition of rank than the one from Labourie [18].

Given a convex map from X to $\mathbb{P}(\mathbb{R}^d)$ one can define a cross ratio on X $\mathsf{b}^\xi : X^{(4)} \to \mathbb{R}$ by

$$\mathsf{b}^\xi(x, y, z, t) = \frac{\varphi(u)}{\psi(u)} \frac{\psi(v)}{\varphi(v)} \tag{10}$$

where $\varphi \in \xi^*(x)$, $\psi \in \xi^*(z)$, $u \in \xi(y)$ and $v \in \xi(t)$. Remark that the result does not depend on the choice of φ, ψ, u and v made.

Remark 6.5 (Labourie [18])**.** If $\xi(X)$ generates $\mathbb{R}^d$ then b^ξ has rank d.

It turns out that the converse is also true in a stronger version.

Theorem 6.6 (Labourie [18])**.** *Consider a rank d cross ratio $\mathsf{b} : X^{(4)} \to \mathbb{R}$ and assume a group G acts on X leaving b invariant, then there exists a representation $\rho : G \to \mathrm{PGL}(d, \mathbb{R})$ and a ρ-equivariant convex map*

$$(\xi, \xi^*) : X \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*})$$

such that $\xi(X)$ generates $\mathbb{R}^d$ and $\mathsf{b} = \mathsf{b}^\xi$. The representation and the convex map are unique modulo conjugation via $\mathrm{PGL}(d, \mathbb{R})$.

Labourie's original statement is for $X = S^1$ the circle, and $G = \pi_1\Sigma$ the fundamental group of a closed hyperbolic surface, nevertheless the proof of existence is an exact copy of Proposition 5.7 of Labourie [18] and the proof of uniqueness is an exact copy of Lemma 4.3 of Labourie [18].

The main consequence we are interested in is the existence of a linear action of G on $\mathbb{R}^d$ when G acts on X preserving a rank d cross ratio.

We will need the following Lemma for the proof of Hilbert's entropy rigidity.

Lemma 6.7. *Consider a convex map $(\xi, \xi^*) : X \to \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\mathbb{R}^{d^*})$ and the adjoint equivariant maps*

$$(\eta, \eta^*) := (M(\xi, \xi^*), \Phi(\xi, \xi^*)) : X \to \mathbb{P}(\mathfrak{sl}(d, \mathbb{R})) \times \mathbb{P}(\mathfrak{sl}(d, \mathbb{R})^*),$$

defined on Section 4. Then

$$\mathsf{b}^\eta(x, y, z, t) = \mathsf{b}^\xi(x, y, z, t)\mathsf{b}^\xi(y, x, t, z).$$

Proof. The proof is an explicit calculation. □

7. The cross ratio of a Hölder cocycle

Consider now Γ a convex co-compact group of a $\mathrm{CAT}(-1)$ space X. Two Hölder cocycles $c, c' : \Gamma \times \partial_\infty\Gamma \to \mathbb{R}$ are said to be *cohomologous* if there exists a Hölder-continuous function $U : \partial_\infty\Gamma \to \mathbb{R}$ such that for all $\gamma \in \Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma x) - U(x).$$

One easily deduces from the definition that the set of periods $\{\ell_c(\gamma) : \gamma \in \Gamma\}$ of a Hölder cocycle is a cohomological invariant.

Theorem 7.1 (Ledrappier [19])**.** *Two Hölder cocycles are cohomologous if and only if they have the same period for every non torsion $\gamma \in \Gamma$.*

We shall be interested in cocycles whose periods are positive, i.e. such that $\ell_c(\gamma) > 0$ for every non torsion $\gamma \in \Gamma$. The *entropy*† of such cocycle is defined by:

$$h_c = \limsup_{t\to\infty} \frac{1}{t} \log \#\{[\gamma] : \ell_c(\gamma) \le t\} \in \mathbb{R}_+ \cup \{\infty\}.$$

†In [20] this is called the exponential growth rate of the cocycle.

Given a Hölder cocycle there exists a *dual Hölder cocycle* $\overline{c}$ determined by the equation

$$\ell_{\overline{c}}(\gamma) = \ell_c(\gamma^{-1})$$

(see Ledrappier [19] or [20] for details). Denote by

$$\partial_\infty^{(2)}\Gamma = \partial_\infty\Gamma \times \partial_\infty\Gamma - \{(x,x) : x \in \partial_\infty\Gamma\}.$$

A *Gromov product* for the ordered pair $(c,\overline{c})$ is a function $[\cdot,\cdot] = [\cdot,\cdot]_{(c,\overline{c})} : \partial_\infty^{(2)}\Gamma \to \mathbb{R}$ such that for every $\gamma \in \Gamma$ and $x, y \in \partial_\infty\Gamma$ distinct one has

$$[\gamma x, \gamma y] - [x,y] = -(\overline{c}(\gamma,x) + c(\gamma,y))^{\ddagger}.$$

Theorem 7.2 (Ledrappier [19])**.** *Given a Hölder cocycle c there exists a dual cocycle $\overline{c}$. If c has finite and positive entropy then there exists a Gromov product for the pair $(c,\overline{c})$. Moreover, if $y \to x$ then $[x,y] \to -\infty$.*

We will hence adopt the convention $[x,x] = -\infty$.

Lemma 7.3. *Let $\{c,\overline{c}\}$ be a pair of dual Hölder cocycles with positive and finite entropy, and $[\cdot,\cdot]$ a Gromov product for the ordered pair $(c,\overline{c})$. Then the function $\mathsf{b}_c : \partial_\infty\Gamma^{(4)} \to \mathbb{R}$ defined by*

$$\mathsf{b}_c(x,y,z,t) = \exp\{[x,y] - [z,y] + [z,t] - [x,t]\},$$

is a Γ-invariant cross ratio on $\partial_\infty\Gamma$. Moreover, b_c only depends on the cohomology class of c. In particular the cross ratio b_c is uniquely determined by the periods $\{\ell_c(\gamma) : \gamma \in \Gamma\}$ of c.

Hence, we define the *cross ratio*[§] associated to a Hölder cocycle c as the function b_c of the last Lemma. Remark that by definition $\mathsf{b}_c \geq 0$.

Proof. The properties from the definition 6.1 are easily verified and Γ-invariance follows from the definition of Gromov product.

Let us focus on the fact that b_c only depends on the cohomology class of c. Remark that Ledrappier's Theorem 7.1 implies that the cohomology class of $\overline{c}$ is determined by the cohomology class of c. Consider then c' a cocycle cohomologous to c, i.e. $c'(\gamma,x) - c(\gamma,x) = U(\gamma x) - U(x)$. The Gromov product associated to the ordered pair $(c',\overline{c})$ is $[x,y]' = [x,y] + U(y)$. One then has

$$[x,y]' - [z,y]' = [x,y] + U(y) - [z,y] - U(y) = [x,y] - [z,y].$$

Hence, the cross ratios associated to $(c,\overline{c})$ and $(c',\overline{c})$ coincide. analog reasoning works for a cocycle cohomologous to $\overline{c}$. The last statement follows from this and Ledrappier's Theorem 7.1. $\square$

It would be nice to have Benoist-like formula (see Benoist [3, Lemma 1.6]):

Question. Let c be a Hölder cocycle with finite and positive entropy, is it true that for every non torsion $\gamma, h \in \Gamma$ one has

$$\exp\{\ell_c(\gamma^n h^n) - \ell_c(\gamma^n) - \ell_c(h^n)\} \to \mathsf{b}_c(\gamma_-, h_+, h_-, \gamma_+)$$

as $n \to \infty$?

---

‡ Remark this definition applyed to a CAT(-1)-space gives the opposite of the usual Gromov product.

§ This is Ledrappier's [19] construction of a cross ratio, but in our case the Hölder cocycle is not necessarily what he calls pair.

The following Lemma clarifies the relation between the cross ratios b_c and $\mathsf{b}_{\overline{c}}$. I would like to thank Qiongling Li for discussions concerning this matter.

Lemma 7.4. *Consider a Hölder cocycle c with finite and positive entropy. Then for every $(x,y,z,t)\in\partial_\infty\Gamma^{(4)}$ one has*

$$\mathsf{b}_c(x,y,z,t)=\mathsf{b}_{\overline{c}}(y,x,t,z)$$

and

$$\mathsf{b}_{c+\overline{c}}(x,y,z,t)=\mathsf{b}_c(x,y,z,t)\mathsf{b}_{\overline{c}}(x,y,z,t).$$

Proof. Let $[\cdot,\cdot]_1$ be the Gromov product for the ordered pair $(c,\overline{c})$. An explicit computation shows that $[\cdot,\cdot]_2:\partial_\infty^{(2)}\Gamma\to\mathbb{R}$ defined by

$$[x,y]_2=[y,x]_1$$

is a Gromov product for the ordered pair $(\overline{c},c)$. Hence the first equality in the Lemma. The second equality follows then by remarking that $[x,y]_3=[x,y]_1+[x,y]_2$ is a Gromov product for $(c+\overline{c},c+\overline{c})$.

□

8. Convex representations and cross ratios

Consider now a convex representation $\rho:\Gamma\to\mathrm{PGL}(d,\mathbb{R})$ with equivariant maps $\xi:\partial_\infty\Gamma\to\mathbb{P}(\mathbb{R}^d)$ and $\xi^*:\partial_\infty\Gamma\to\mathbb{P}(\mathbb{R}^{d^*})$. Lemma 4.3 implies that (ξ,ξ^*) is a convex map in the sense of Section 6, we define then the *cross ratio of* ρ to be the cross ratio of the equivariant map ξ, i.e.

$$\mathsf{b}_\rho=\mathsf{b}^\xi.$$

Consider now the Hölder cocycles $\beta_\rho,\overline{\beta}_\rho:\Gamma\times\partial_\infty\Gamma\to\mathbb{R}$ defined on Section 4. Lemma 4.4 implies that $\{\beta_\rho,\overline{\beta}_\rho\}$ is a pair of dual cocycles.

Consider now the function $\mathscr{G}:\mathbb{P}(\mathbb{R}^{d^*})\times\mathbb{P}(\mathbb{R}^d)-\{(\varphi,v):\varphi(v)=0\}\to\mathbb{R}$ defined by

$$\mathscr{G}(\theta,v)=\log\frac{|\theta(v)|}{\|\theta\|\|v\|}$$

and define $[\cdot,\cdot]:\partial_\infty^{(2)}\Gamma\to\mathbb{R}$ by

$$[x,y]=\mathscr{G}(\xi^*(x),\xi(y)).$$

A direct computation shows that the function $[\cdot,\cdot]:\partial_\infty^{(2)}\Gamma\to\mathbb{R}$ is a Gromov product for the ordered pair $(\beta_\rho,\overline{\beta}_\rho)$. The cross ratio associated to the cocycle β_ρ is

$$\mathsf{b}_{\beta_\rho}(x,y,z,t)=\left|\frac{\varphi(u)}{\psi(u)}\frac{\psi(v)}{\varphi(v)}\right|$$

where $\varphi\in\xi^*(x)$, $\psi\in\xi^*(z)$, $u\in\xi(y)$ and $v\in\xi(t)$. Hence $\mathsf{b}_{\beta_\rho}=|\mathsf{b}_\rho|$. Remark also that the cross ratio associated to A_ρ, the irreducible adjoint representation of ρ, verifies

$$|\mathsf{b}_{\mathsf{A}_\rho}|=\mathsf{b}_{\beta_\rho+\overline{\beta}_\rho}.$$

As a conclusion of this section we state the following consequence of Lemma 7.3:

Lemma 8.1. *Let $\rho:\Gamma\to\mathrm{PSL}(d,\mathbb{R})$ be an irreducible convex representation, then the cross ratio $|\mathsf{b}_\rho|$ is induced by the Hölder cocycle β_ρ, and is thus uniquely determined by $\{\lambda_1(\rho\gamma):\gamma\in\Gamma\}$. Moreover the cross ratio $|\mathsf{b}_{\mathsf{A}_\rho}|$ is induced by the Hölder cocycle $\beta_\rho+\overline{\beta}_\rho$ and is thus uniquely determined by $\{\lambda_1(\rho\gamma)-\lambda_d(\rho\gamma):\gamma\in\Gamma\}$.*

9. Proximal Representations of semi-simple Lie groups

Consider a real algebraic non compact semi-simple Lie group G. Let K be a maximal compact subgroup of G and τ the Cartan involution on $\mathfrak{g}$ whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p}=\{v\in\mathfrak{g}:\tau v=-v\}$ and $\mathfrak{a}$ a maximal abelian subspace contained in $\mathfrak{p}$.

Let Σ be the set of (restricted) roots of $\mathfrak{a}$ on $\mathfrak{g}$, $\mathfrak{a}^+$ a closed Weyl chamber, Σ^+ a system of positive roots on Σ associated to $\mathfrak{a}^+$ and denote by Π the set of simple roots associated to the choice Σ^+.

For $g\in G$ denote by $\lambda(g)\in\mathfrak{a}^+$ its *Jordan projection*, this is the unique element on $\mathfrak{a}^+$ such that $\exp\lambda(g)$ is conjugated to the $\mathbb{R}$-regular element on the Jordan decomposition of g.

For an irreducible representation $\phi: G\to \mathrm{PGL}(d,\mathbb{R})$ denote by $\chi_\phi\in\mathfrak{a}^*$ its *restricted highest weight*. For every $g\in G$ one has, by definition,

$$\lambda_1(\phi g)=\chi_\phi(\lambda(g)). \tag{11}$$

We say that ϕ is *proximal* if $\phi(G)$ contains a proximal element.One has the following standard proposition in Representation Theory.

Proposition 9.1 (see Benoist [5, Section 2.2])**.** *The set of restricted weights of $\mathfrak{a}^*$ is in bijection with (equivalence classes of) irreducible proximal representations of G.*

Let W be the Weyl group of Σ and denote $u_0:\mathfrak{a}\to\mathfrak{a}$ the biggest element in W, u_0 is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i}:\mathfrak{a}\to\mathfrak{a}$ is the defined by $\mathrm{i}:=-u_0$.

Consider $\{\omega_\theta\}_{\theta\in\Pi}$ the set of fundamental weights of Π. We will need the following result of Tits [23].

Proposition 9.2 (Tits [23])**.** *For each $\theta\in\Pi$ there exists a finite dimensional proximal irreducible representation $\Lambda_\theta: G\to\mathrm{PGL}(V_\theta)$ such that the restricted highest weight χ_θ of Λ_θ is an integer multiple of the fundamental weight ω_θ.*

We will now specify on the group $\mathrm{Isom}\,\mathbb{H}^k$. The Cartan subspace $\mathfrak{a}_{\mathbb{H}^k}$ is 1-dimensional and is thus identified with $\mathbb{R}$. The *Jordan projection* of $\gamma\in\mathrm{Isom}\,\mathbb{H}^k$ is

$$\lambda_{\mathbb{H}^k}(\gamma)=\inf_{p\in\mathbb{H}^k} d_{\mathbb{H}^k}(p,\gamma p),$$

and thus $\lambda_{\mathbb{H}^k}(\gamma)$ coincides with the translation length $|\gamma|$ when γ is a hyperbolic element.

Remark 9.3. Remark that if $\rho:\mathrm{Isom}\,\mathbb{H}^k\to\mathrm{PGL}(k+1,\mathbb{R})$ is the Klein model of $\mathbb{H}^k$ and $\gamma\in\mathrm{Isom}\,\mathbb{H}^k$ is hyperbolic then $\lambda_1(\rho\gamma)=|\gamma|$ and $\lambda_1(\mathrm{Ad}\,\rho\gamma)=2|\gamma|$.

10. Computing the rank of ${\mathbb{B}_{\mathbb{H}}}^\alpha$

Consider $\phi:\mathrm{Isom}\,\mathbb{H}^k\to\mathrm{PGL}(k+1,\mathbb{R})$ the Klein model of $\mathbb{H}^k$. As explained before we have a convex map

$$(\xi,\xi^*):\partial_\infty\mathbb{H}^k\to\mathbb{P}(\mathbb{R}^{k+1})\times\mathbb{P}({\mathbb{R}^{k+1}}^*).$$

Define the cross ratio $\mathbb{B}_{\mathbb{H}^k}$ on $\partial_\infty\mathbb{H}^k$ by[†]

$$\mathbb{B}_{\mathbb{H}^k}=\mathsf{b}^\xi.$$

[†]For $k=2$ this is the square of the usual cross ratio on the circle.

Remark 10.1. Remark that $\mathbb{B}_{\mathbb{H}^k}$ is non negative: this is obvious for $k \geq 3$ and for $k = 2$ one uses the fact that $\xi(\partial_\infty \mathbb{H}^2) \subset \mathbb{P}(\mathbb{R}^3)$ is the boundary of a convex set. Remark also that the rank of $\mathbb{B}_{\mathbb{H}^k}$ is $k+1$.

As observed before, if α is a positive real number then ${\mathbb{B}_{\mathbb{H}^k}}^\alpha$ is also a cross ratio.

Remark 10.2. Let Γ be a Zariski dense convex co-compact subgroup of $\operatorname{Isom}\mathbb{H}^k$ and α a positive real number, then the cross ratio ${\mathbb{B}_{\mathbb{H}^k}}^\alpha | \mathrm{L}_\Gamma$ is induced by a Hölder cocycle (namely α·Busseman's cocycle) and hence it is uniquely determined by the periods of α·Busseman, i.e. by the set $\{\alpha|\gamma| : \gamma \in \Gamma\}$.

Deciding whether the cross ratio ${\mathbb{B}_{\mathbb{H}^k}}^\alpha$ has finite rank can be a difficult task, the actual definition is hard to handle.

The purpose of this section is to show that, in order to compute the rank of ${\mathbb{B}_{\mathbb{H}^k}}^\alpha$ one only needs to check on the limit set of a Zariski dense subgroup of $\operatorname{Isom}\mathbb{H}^k$. This is a key step in the proofs of Theorems B and D.

Theorem 10.3. *Consider a Zariski dense subgroup Γ of $\operatorname{Isom}\mathbb{H}^k$ and a positive real number α. If the cross ratio ${\mathbb{B}_{\mathbb{H}^k}}^\alpha | \mathrm{L}_\Gamma : \mathrm{L}_\Gamma^{(4)} \to \mathbb{R}$ has finite rank, then*

$$\operatorname{rank} {\mathbb{B}_{\mathbb{H}^k}}^\alpha = \operatorname{rank} {\mathbb{B}_{\mathbb{H}^k}}^\alpha | \mathrm{L}_\Gamma \, .$$

10.1. **Polynomial functions on the Furstenberg boundary.** We will freely use the notations of Section 9. Let P be a minimal parabolic subgroup of G and denote by $\mathscr{F} = G/P$ the *Furstenberg boundary* of the symmetric space of G. Consider Δ a Zariski dense subgroup of G. We have the following Proposition of Benoist [2].

Proposition 10.4 (Benoist [2])**.** *The action of Δ on $\mathscr{F}$ has a unique minimal closed invariant set.*

This smallest closed invariant set is called *the limit set* of Δ and denoted by L_Δ. We need the following technical Proposition:

Proposition 10.5. *Consider Δ a Zariski dense subgroup of G. Consider also a finite family of polynomial functions $P_i : \mathscr{F} \to \mathbb{R}$ for $i \in \{1, \ldots, k\}$ and positive real numbers α_i. Consider*

$$f := \sum_1^k \varepsilon_i |P_i|^{\alpha_i} : \mathscr{F} \to \mathbb{R}$$

where $\varepsilon_i \in \{1, -1\}$ and assume that f vanishes on L_Δ, then f vanishes on $\mathscr{F}$.

The proof of the Proposition goes by adapting arguments from Section 7 of Benoist [2]. We reproduce here these arguments for completeness. We would like to thank Yves Benoist for discussions that lead to this proof.

We need the following Lemmas from Benoist [2].

Lemma 10.6 (Benoist [2, Lemma 7.1])**.** *Consider a finite family of 1-parameter subgroups $\{g_i^t : t \in \mathbb{R}\}$ $i \in \{1, \ldots, s\}$ of G. Consider the semi-group H generated by*

$$\{g_i^t : i \in \{1, \ldots, s\} \text{ and } t \in [1, \infty)\}.$$

If H is Zariski dense in G then H has non empty interior.

An element $g \in G$ is called *semi-simple* if the unipotent element in its Jordan decomposition is trivial.

Lemma 10.7 (Benoist [2])**.** *Let Δ be a Zariski dense subgroup of G. Then there exists a subgroup Δ' of Δ such that*

- *Δ' is still Zariski dense on G,*
- *every element of Δ' is semi-simple.*

10.2. **Hardy fields.** Consider $\mathcal{K}$ the ring of germs at infinity of C^∞ functions defined on a half line $[t_0,\infty)$. A *Hardy field* is a sub-field of $\mathcal{K}$ stable under derivation. Our interest in considering Hardy fields is that if f belongs to a Hardy field then $1/f$ is well defined and hence f has no zeros on some half line $[t,\infty)$.

Proposition 10.8 (Benoist [2])**.** *Let $\mathcal{H}$ be a Hardy field. Let α be a positive real number and $f\in\mathcal{H}$ then there exists a Hardy field $\mathcal{H}'$ that contains $\mathcal{H}$ and $|f|^\alpha$.*

Recall that G is a real algebraic semi-simple Lie group and that $\mathscr{F}$ is its Furstenberg boundary. Consider a finite family of polynomial functions $P_i:\mathscr{F}\to\mathbb{R}$ for $i\in\{1,\dots,k\}$ and positive real numbers α_i. Consider

$$f=\sum_1^k\varepsilon_i|P_i|^{\alpha_i}:\mathscr{F}\to\mathbb{R}$$

where $\varepsilon_i\in\{1,-1\}$

Lemma 10.9. *Consider the 1-parameter group $\{g^t:t\in\mathbb{R}\}$ generated by an $\mathbb{R}$-regular element $g\in G$ and fix $x_0\in\mathscr{F}$. Consider also $h_1,h_2\in G$. Then the function*

$$t\mapsto f(h_2g^th_1\cdot x_0)$$

either has finite zeros or identically vanishes on some half line.

Proof. For each polynomial P_i, the function

$$t\mapsto P_i(h_2g^th_1\cdot x_0)$$

is a linear combination of functions of the form $t\mapsto e^{kt}$ for some $k\in\mathbb{R}$ (this is consequence of g being $\mathbb{R}$-regular). All these functions belong to a Hardy field, hence using Proposition 10.8, so does $t\mapsto|P_i(h_2g^th_1\cdot x_0)|^{\alpha_i}$ and thus so does $t\mapsto f(h_2g^th_1\cdot x_0)$. $\square$

10.3. **Proof of Proposition 10.5.** We need the following Lemma.

Lemma 10.10 (Benoist [2, Lemma 7.3])**.** *Let V be a real analytic manifold, $\psi: V\times\mathbb{R}\to\mathbb{R}$ an analytic function. Consider $Z=\psi^{-1}(0)$ and let $p:Z\to V$ be the projection on the first coordinate. Assume that $p^{-1}(m)$ is finite for all $m\in V$, then there exists a non empty open set U of V such that $p^{-1}(\overline{U})$ is compact on Z.*

Proof of Proposition 10.5. Fix a point $x_0\in\mathrm{L}_\Delta$ and consider the function $\Phi:G\to\mathbb{R}$ defined by

$$\Phi(g)=f(g\cdot x_0).$$

Remark that since L_Δ is Δ-invariant and f vanishes on L_Δ one has $\Phi(\Delta)\equiv0$.

Consider the subgroup Δ' of semi-simple elements of Δ given by Lemma 10.7. Consider H the semi-group generated by $\{g^t:g\in\Delta'\ \theta\geq1\}$. Lemma 10.6 implies that H has non empty interior. We will show that Φ vanishes on H, and hence, since Φ is analytic, it will be constant equal to zero.

An element in H is of the form $h_1\cdots h_s$ where, for each i, one has $h_i=g_i^{t_i}$ for some $g_i\in\Delta'$ and $t_i\in[1,\infty)$. To show that Φ vanishes on H we will proceed by induction, i.e. we will show that if $h_1,h_2\in H$ and $g\in\Delta'$ are such that

$\Phi(h_2 g^n h_1) = 0$ for every $n \in \mathbb{N}$ then $\Phi(h_2 g^t h_1) = 0$ for all $t \in [1, \infty)$, since $\Phi(\Delta') = 0$ the result will follow.

For $g \in \Delta'$ write $g = m_g a_g$ where m_g is elliptic and a_g is $\mathbb{R}$-regular and consider the one parameter subgroup $g^t = m_g^t a_g^t$. Let M be the closure of the group

$$\{m_g^t : t \in \mathbb{R}\}.$$

Remark that M is a compact group and hence it is Zariski closed. Consider then

$$Z = \{(m, t) \in M \times [1, \infty) : \Phi(h_2 m a_g^t h_1) = 0\}$$

and $p : Z \to M$ the projection on the first coordinate.

Remark that for every $m \in M$ one has that $p^{-1}(m)$ is either $m \times [1, \infty)$ or finite. Indeed, Lemma 10.9 states that $t \mapsto \Phi(h_2 m a_g^t h_1)$ belongs to a Hardy field, hence it has finite zeros or it is constant equal to zero.

Assume by contradiction that $Z \neq M \times [1, \infty)$, then there is an open set V of M such that $p^{-1}(m_0)$ is finite for every $m_0 \in V$. Applying Lemma 10.10 implies the existence of an open subset $U \subset V$ such that $p^{-1}(\overline{U})$ is compact, but the set

$$\{n \in \mathbb{N} : m_g^n \in U\}$$

is infinite and, by hypothesis $\Phi(h_2 m_g^n a_g^n h_1) = 0$, hence $(m_g^n, n) \in p^{-1}(\overline{U})$. This contradicts the fact that $p^{-1}(\overline{U})$ is compact. This finishes the proof. □

10.4. **Proof of Theorem 10.3.** Let $p_0 \in \mathbb{N}$ be the rank of $\mathbb{B}_{\mathbb{H}^k}{}^{\alpha}|\,\mathrm{L}_\Gamma$, this is to say, for every $p > p_0$ the function

$$\chi^p(e, u) := \chi^p_{\mathbb{B}_{\mathbb{H}^k}{}^{\alpha}|\,\mathrm{L}_\Gamma}(e, u) = \det_{i,j>0}\left(\mathbb{B}_{\mathbb{H}^k}(e_i, u_j, e_0, u_0)^{\alpha}\right) \tag{12}$$

vanishes identically for every pair of $p+1$-tuples $(e, u) = (e_0, \dots, e_p, u_0, \dots, u_p)$ of points on L_Γ^{p*}. We will show that it also vanishes for every pair of $p+1$-tuples of points on $\partial_\infty \mathbb{H}^k$. Since $\chi^{p_0}_{\mathbb{B}_{\mathbb{H}^k}{}^{\alpha}|\,\mathrm{L}_\Gamma}$ does not vanish this will finish the proof.

Remark that fixing three points in $\partial_\infty \mathbb{H}^k$ and considering $\mathbb{B}_{\mathbb{H}^k}{}^{\alpha}$ as a function on the remaining variable one obtains a rational function on $\partial_\infty \mathbb{H}^k$ to the power α. Consider then common denominator in equation (12) and denote by $Q(e, u)$ the numerator. Remark that $Q(e, u)$ is a sum and subtraction of polynomial functions, each polynomial to a power α.

Fix $\overline{e} = (e_0, \dots, e_p)$ p points in L_Γ and $(u_0, \dots, u_{p+1})$ $p+1$ points in L_Γ. Consider now the function $f(x) = Q((x, \overline{e}), u) : \partial_\infty \mathbb{H}^k \to \mathbb{R}$. The function f vanishes identically on L_Γ, hence Proposition 10.5 states that $f \equiv 0$. Repeating this procedure one can pull out one by one the points on L_Γ to $\partial_\infty \mathbb{H}^k$ to show that $\chi^p_{\mathbb{B}_{\mathbb{H}^k}{}^{\alpha}}$ vanishes identically on $(\partial_\infty \mathbb{H}^k)^{p*}$. This finishes the proof.

11. Proofs of Theorem B, Corollary 1.5 and Corollary 1.6

We will first prove rigidity for the spectral entropy; Hilbert entropy rigidity and Corollary 1.6 follow similarly with minor arrangements. Throughout this section Γ is a Zariski dense convex co-compact group of $\mathbb{H}^k$, $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is an irreducible convex representation and $\alpha = \alpha_\rho$ is the best Hölder exponent of the equivariant map $\xi : \mathrm{L}_\Gamma \to \mathbb{P}(\mathbb{R}^d)$.

11.1. **Proof of Spectral entropy rigidity.** Assume the following equality holds

$$\alpha h_\rho = h_\Gamma \tag{13}$$

and assume that $\mathsf{b}_\rho \geq 0$. Remark 5.2 states that there exists $c > 0$ such that for every non torsion $\gamma \in \Gamma$ one has

$$\lambda_1(\rho\gamma) = c|\gamma|.$$

This equality together with equality (13) imply in fact that

$$\lambda_1(\rho\gamma) = \alpha|\gamma|. \tag{14}$$

The cocycles β_ρ and α·Busseman: $\Gamma \times \mathrm{L}_\Gamma \to \mathbb{R}$ are hence cohomologous, and thus Lemma 7.3 implies their cross ratios coincide, i.e.: $\mathsf{b}_\rho = |\mathsf{b}_\rho| = \mathbb{B}_{\mathbb{H}^k}{}^\alpha | \mathrm{L}_\Gamma$.

Since ρ is irreducible, the cross ratio b_ρ has rank d. Hence proposition 10.3 implies that the cross ratio $\mathbb{B}_{\mathbb{H}^k}{}^\alpha$, defined on all $\partial_\infty \mathbb{H}^k$, also has rank d. Since $\mathbb{B}_{\mathbb{H}^k}{}^\alpha$ is invariant under the full isometry group $\operatorname{Isom}\mathbb{H}^k$, Labourie's Theorem 6.6 applies and uniqueness implies that $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ extends to $\overline{\rho} : \operatorname{Isom}\mathbb{H}^k \to \mathrm{PGL}(d, \mathbb{R})$.

The equivariant map $\xi : \mathrm{L}_\Gamma \to \mathbb{P}(\mathbb{R}^d)$ then comes from a proximal irreducible morphism $\overline{\rho} : \operatorname{Isom}\mathbb{H}^k \to \mathrm{PGL}(d, \mathbb{R})$ and is thus of class C^∞, in particular it is Lipschitz i.e. $\alpha = 1$. Hence $\mathsf{b}_{\overline{\rho}} = \mathbb{B}_{\mathbb{H}^k}$, therefore $d = \operatorname{rank} \mathsf{b}_{\overline{\rho}} = \operatorname{rank} \mathbb{B}_{\mathbb{H}^k} = k + 1$. Again Labourie's Theorem 6.6 implies that $\overline{\rho}$ is the Klein model of $\mathbb{H}^k$.

11.2. **Hilbert entropy rigidity.** Suppose now that the following equality holds

$$\alpha \mathsf{H}_\rho = h_\Gamma.$$

Similar reasoning to spectral entropy rigidity yields

$$\lambda_1(\rho\gamma) - \lambda_d(\rho\gamma) = 2\alpha|\gamma|$$

for every non torsion $\gamma \in \Gamma$. Consider the irreducible adjoint representation $\mathsf{A}_\rho : \Gamma \to \mathrm{PGL}(V_\rho)$ (recall Lemma 4.7). Again, analogue reasoning to the subsection 11.1 implies that $|\mathsf{b}_{\mathsf{A}_\rho}| = \mathbb{B}_{\mathbb{H}^k}{}^{2\alpha} | \mathrm{L}_\Gamma$.

Since $\mathsf{b}_\rho \geq 0$ Lemma 6.7 implies that so is $\mathsf{b}_{\mathsf{A}_\rho}$, hence

$$\mathsf{b}_{\mathsf{A}_\rho} = \mathbb{B}_{\mathbb{H}^k}{}^{2\alpha} | \mathrm{L}_\Gamma.$$

As before, the cross ratio $\mathbb{B}_{\mathbb{H}^k}{}^{2\alpha}$ has finite rank and thus $\mathsf{A}_\rho : \Gamma \to \mathrm{PGL}(V_\rho)$ extends to a representation $\overline{\mathsf{A}}_\rho : \operatorname{Isom}\mathbb{H}^k \to \mathrm{PGL}(V_\rho)$. Hence $\alpha = 1$ and highest restricted weight $\chi_{\overline{\mathsf{A}}_\rho}$ of $\overline{\mathsf{A}}_\rho$ is

$$\chi_{\overline{\mathsf{A}}_\rho}(\lambda_{\mathbb{H}^k}(\gamma)) = 2|\gamma|.$$

Since $\overline{\mathsf{A}}_\rho$ is proximal and irreducible the last equations implies (using Remark 9.3) that $\overline{\mathsf{A}}_\rho$ is the adjoint representation of $\mathrm{PSO}(k, 1)$,

$$\overline{\mathsf{A}}_\rho = \mathrm{Ad} : \mathrm{PSO}(1, k) \to \mathrm{PGL}(\mathfrak{so}(1, k)).$$

This finishes the proof.

11.3. **Proof of Corollary 1.6.** Assume now that Γ acts co-compactly on $\mathbb{H}^2$ and that the equality

$$\alpha h_\rho = 1 \tag{15}$$

holds.

Similarly to Subsection 11.1, $\lambda_1(\rho\gamma) = \alpha|\gamma|$, for every non torsion $\gamma \in \Gamma$, and the cross ratios $|\mathsf{b}_\rho|$ and ${\mathbb{B}_{\mathbb{H}^2}}^\alpha$ coincide. Hence $|\mathsf{b}_\rho|$ is invariant under group $\mathrm{PSL}(2,\mathbb{R})$. The same happens then for the cross ratio b_ρ (see Lemma 6.2). Labourie's Theorem 6.6 applies and uniqueness implies that $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ extends to $\overline{\rho} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(d,\mathbb{R})$. Hence α equals 1.

Let χ_ρ be the restricted highest weight of the irreducible representation $\overline{\rho} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(d,\mathbb{R})$. Equation (11) states that

$$\chi_\rho(\lambda_{\mathbb{H}^2}(\gamma)) = \lambda_1(\rho\gamma) = |\gamma|.$$

Proposition 9.1 implies that $\overline{\rho}$ is the Klein model of $\mathbb{H}^2$. This finishes the proof.

12. Hyperconvex representations and Theorem C

We will freely use the notations of Section 9. Let G be a real non compact semi-simple Lie group and denote by $\mathscr{F}$ the *Furstenberg boundary* of the symmetric space of G. The product $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit denoted by $\mathscr{F}^{(2)}$.

Definition 12.1. A representation $\rho : \Gamma \to G$ is *hyperconvex* if there exists a ρ-equivariant Hölder-continuous map $\zeta : \partial_\infty\Gamma \to \mathscr{F}$ such that if $x \neq y$ are distinct points in $\partial_\infty\Gamma$ then the pair $(\zeta(x), \zeta(y))$ belongs to $\mathscr{F}^{(2)}$.

The following Lemma relates hyperconvex representations to convex ones.

Lemma 12.2. *If $\rho : \Gamma \to G$ is Zariski dense and hyperconvex, and $\Lambda : G \to \mathrm{PGL}(V)$ is a finite dimensional irreducible proximal representation, then the composition $\Lambda \circ \rho : \Gamma \to \mathrm{PGL}(V)$ is irreducible and convex.*

Proof. A proximal representation $\Lambda : G \to \mathrm{PGL}(V)$ induces a C^∞ equivariant map $: \mathscr{F} \to \mathbb{P}(V)$. Considering the dual representation $\Lambda^* : G \to \mathrm{PGL}(V^*)$ one obtains another equivariant map $: \mathscr{F} \to \mathrm{PGL}(V^*)$. The remainder of the statement is direct. $\square$

We need the following Theorem from [20].

Theorem 12.3 ([20, Corollary 7.6])**.** *Let $\rho : \Gamma \to G$ be a Zariski dense hyperconvex representation, then there exists a Γ-invariant Hölder-continuous function $F : \mathsf{U}\Gamma \to \mathfrak{a}$ such that for every non torsion conjugacy class $[\gamma] \in [\Gamma]$ one has*

$$\int_{[\gamma]} F = \lambda(\rho\gamma).$$

Moreover, if $\varphi|\mathfrak{a}^+ - \{0\} > 0$ then $\varphi(F) : \mathsf{U}\Gamma \to \mathbb{R}$ is Livšic-cohomologous to a positive function.

Fix an action of Γ on CAT(-1) space X, and denote L_Γ its limit set on $\partial_\infty X$. Assume from now on that $\rho : \Gamma \to G$ is a Zariski dense hyperconvex representation and denote by α the Hölder exponent of the equivariant map $\zeta : \mathrm{L}_\Gamma \to \mathscr{F}$.

Lemma 12.4. *For every simple root $\theta \in \Pi$ and every non torsion $\gamma \in \Gamma$ one has*

$$\alpha \leq \frac{\theta(\lambda(\rho\gamma))}{|\gamma|}.$$

Proof. Consider $\Lambda_\theta\circ\rho:\Gamma\to\mathrm{PGL}(V_\theta)$ the irreducible convex representation given by Tits's Proposition 9.2 and Lemma 12.2. One then has

$$\theta(\lambda(\rho\gamma))=\lambda_1(\Lambda_\theta\circ\rho\gamma)-\lambda_2(\Lambda_\theta\circ\rho\gamma).$$

The Lemma follows from Lemma 4.11. □

12.1. **Proof of Theorem C.** The proof is very similar to the proof of Theorem A. Consider $F:\mathsf{U}\Gamma\to\mathfrak{a}$ given by Theorem 12.3, i.e. F verifies

$$\int_{[\gamma]}F=\lambda(\rho\gamma)$$

for every non torsion conjugacy class $[\gamma]$ of Γ.

Let φ be a linear functional such that $\varphi|\mathfrak{a}^+-\{0\}>0$. Using Theorem 12.3 we may assume that $f=\varphi(F):\mathsf{U}\Gamma\to\mathbb{R}_+^*$ a positive function. Remark that

$$\varphi(\lambda(\rho\gamma))=\int_{[\gamma]}f$$

and hence $h_f=h_\varphi$. Consider a sequence of conjugacy classes $[\gamma_n]$ such that

$$\frac{\mathrm{Leb}_{\gamma_n}}{|\gamma_n|}\to m_{-h_\varphi f}$$

as $n\to\infty$, where $m_{-h_\varphi f}$ denotes the equilibrium state of $-h_\varphi f$. Analog reasoning to Theorem A shows that

$$\frac{\varphi(\lambda(\rho\gamma_n))}{|\gamma_n|}\leq\frac{h(\phi,m_{-h_\varphi f})}{h_\varphi}(1+\varepsilon),$$

and thus, using Lemma 12.4 one finds that

$$\frac{\alpha h_\varphi}{h_\Gamma}\leq\frac{\alpha h_\varphi}{h(\phi,m_{-h_\varphi f})}\leq\frac{\theta(\lambda(\rho\gamma_n))}{\varphi(\lambda(\rho\gamma_n))}(1+\varepsilon)$$

for every simple root $\theta\in\Pi$. We now try to maximize the function $\mathsf{V}:\mathbb{P}(\mathfrak{a}^+)\to\mathbb{R}$

$$\mathsf{V}(a)=\min_{\theta\in\Pi}\left\{\frac{\theta(a)}{\varphi(a)}\right\}.$$

We need the following Linear Algebra Lemma. Consider an n-dimensional vector space W, a k-*simplex* is the convex hull of $k+1$ points $\{x_0,\dots,x_k\}$ in W such that for every $i\in\{0,\dots,k\}$ the set $\{x_0,\dots,x_k\}-\{x_i\}$ is linearly independent.

Lemma 12.5. *Consider $n+1$ affine linear functionals $\varphi_i:W\to\mathbb{R}$ on an n-dimensional vector space V, such that*

$$\Delta:=\bigcap_0^n\{v\in W:\varphi_i(v)\geq0\}$$

is an n-dimensional simplex. Then

$$\max_{v\in\Delta}\min\{\varphi_i(v):i\in\{0,\dots,n\}\}$$

is given in the point all the φ_i's coincide, i.e. in the unique $v\in\Delta$ such that

$$\varphi_0(v)=\varphi_1(v)=\cdots=\varphi_n(v).$$

Proof. The proof of the Lemma is trivial. □

Fix now some vector v in the interior of $\mathfrak{a}^+$ such that $\varphi(v) \neq 0$ and consider the map $T : \ker\varphi \to \mathbb{P}(\mathfrak{a})$ given by $w \mapsto \mathbb{R}(v+w)$, this map identifies $\ker\varphi$ with $\mathbb{P}(\mathfrak{a}) - \mathbb{P}(\ker\varphi)$. Via this mapping one gets that the functions $T_\theta : \ker\varphi \to \mathbb{R}$ given by

$$T_\theta(w) := \frac{\theta(w+v)}{\varphi(w+v)} = \frac{\theta(v)}{\varphi(v)} + \frac{\theta(w)}{\varphi(v)}$$

are affine functionals. Since φ is positive on the Weyl chamber $\mathfrak{a}^+ - \{0\}$ we get that

$$\Delta = T^{-1}(\mathbb{P}(\mathfrak{a}^+)) = T^{-1}(\mathbb{P}(\bigcap_{\theta\in\Pi}\{\theta \geq 0\})) = \bigcap_{\theta\in\Pi}\{T_\theta \geq 0\}$$

is a simplex of dimension $\dim\mathfrak{a} - 1 = \dim\ker\varphi$.

Remark that $\mathsf{V} \circ T = \min\{T_\theta : \theta \in \Pi\}$, hence Lemma 12.5 implies that the maximum of $\mathsf{V} \circ T|\Delta$ is realized where all the functions $\{T_\theta : \theta \in \Pi\}$ coincide, i.e. in the set

$$\{a \in \mathfrak{a}^+ : \theta_1(a) = \theta_2(a) \text{ for every pair } \theta_1, \theta_2 \in \Pi\}.$$

This is exactly the barycenter of the Weyl chamber $\mathsf{bar}_{\mathfrak{a}^+}$.

Hence

$$\frac{\alpha h_\varphi}{h_\Gamma} \leq \frac{\alpha h_\varphi}{h(\phi, m_{-h_\varphi f})} \leq \mathsf{V}(\lambda(\rho\gamma_n))(1+\varepsilon) \leq \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\varphi(\mathsf{bar}_{\mathfrak{a}^+})}(1+\varepsilon). \tag{16}$$

This shows the desired inequality.

Remark 12.6. As in Theorem A, Remark that equality in equation (16) implies that f is Livšic cohomologous to a constant.

13. Proofs of Theorem D and Corollary 1.12

Assume now that Γ is a Zariski dense convex-co-compact group of $\mathbb{H}^k$. Consider a hyperconvex representation $\rho : \Gamma \to G$ and consider χ, a restricted weight of G. Suppose the equality

$$\alpha_\rho h_\chi = h_\Gamma \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})}$$

holds, where θ is any simple root.

Remark 12.6 implies that the function $f_\chi = \chi(F) : \mathsf{U}\Gamma \to \mathbb{R}_+^*$ such that

$$\chi(\lambda(\rho\gamma)) = \int_{[\gamma]} f_\chi$$

is Livšic cohomologous to a constant. Consider now the proximal irreducible representation $\Lambda_\chi : G \to \mathrm{PGL}(V_\chi)$ associated to χ by Proposition 9.1, and the composition

$$\phi = \Lambda_\chi \circ \rho : \Gamma \to \mathrm{PGL}(V_\chi).$$

Lemma 12.2 states that ϕ is irreducible and convex and by definition one has

$$\lambda_1(\phi\gamma) = \chi(\lambda(\rho\gamma)) = c|\gamma|.$$

The proof now follows similar to Theorem B. Consider b_ϕ the cross ratio on $\mathrm{L}_\Gamma \subset \partial_\infty\mathbb{H}^k$ of the representation ϕ. The last equality together with Lemma 7.3 imply that $|\mathsf{b}_\phi| = \mathbb{B}_{\mathbb{H}^k}{}^c|\,\mathrm{L}_\Gamma$.

We now distinguish the two situations:

- Assume $\mathsf{b}_\phi \geq 0$ (i.e. we are in the hypothesis of Theorem D): We then have that $\mathsf{b}_\phi = \mathbb{B}_{\mathbb{H}^k}{}^c|\,\mathrm{L}_\Gamma$. Since ϕ is irreducible, the cross ratio b_ϕ has finite rank and Proposition 10.3 implies that the cross ratio $\mathbb{B}_{\mathbb{H}^k}{}^c$, defined on $\partial_\infty \mathbb{H}^k$, has the same rank. Hence Labourie's Theorem 6.6 applies and uniqueness implies that $\phi : \Gamma \to \mathrm{PGL}(V_\chi)$ extends to $\overline{\phi} : \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(V_\chi)$.
- Γ is co-compact ion $\mathbb{H}^2$ (i.e. we are in the hypothesis of Corollary 1.12): In this case we have $|\mathsf{b}_\phi| = \mathbb{B}_{\mathbb{H}^2}{}^c$, and thus $|\mathsf{b}_\phi|$ is invariant under the group $\mathrm{PSL}(2,\mathbb{R})$. The same happens then for the cross ratio b_ϕ (see Lemma 6.2). Labourie's Theorem 6.6 applies and uniqueness implies that $\phi : \Gamma \to \mathrm{PGL}(V_\chi)$ extends to $\overline{\phi} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(V_\chi)$.

In both cases we get that $\phi : \Gamma \to \mathrm{PGL}(V_\chi)$ extends to $\overline{\phi} : \mathrm{Isom}\,\mathbb{H}^k \to \mathrm{PGL}(V_\chi)$. The Zariski closure of $\phi(\Gamma)$ is $\overline{\phi}(\mathrm{Isom}\,\mathbb{H}^k)$ on one hand, and $\Lambda_\chi(G)$ on the other. Hence $\Lambda_\chi(G) = \overline{\phi}(\mathrm{Isom}\,\mathbb{H}^k)$. This finishes the proof.

14. Proof of Corollary 1.13

We will now prove the following Corollary.

Corollary. *Let $\rho : \pi_1\Sigma \to \mathrm{PSL}(d,\mathbb{R})$ be a representation in the Hitchin component and denote by α the best Hölder exponent of the equivariant map $\zeta : \partial_\infty \mathbb{H}^2 \to \mathscr{F}$, then*

$$\alpha h_\rho \leq \frac{2}{d-1} \text{ and } \alpha_\rho \mathsf{H}_\rho \leq \frac{2}{d-1}$$

and either equality holds only if $\rho = \tau_d \circ \mathsf{f}$, where $\tau_d : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$ is the irreducible representation and $\mathsf{f} : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R})$ is the departing action.

The proof goes as follows. Denote by G the Zariski closure of ρ. Guichard's Theorem 14.1 below implies that G is simple, hence $\rho : \pi_1\Sigma \to G$ is again hyperconvex. Consider $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$, and let $\chi \in \mathfrak{a}^*$ be the restricted highest weight of the (irreducible proximal) representation $G \subset \mathrm{PSL}(d,\mathbb{R})$, i.e. if $g \in G$ then $\chi(\lambda(g)) = \lambda_1(g)$.

Remark that by definition the entropy of ρ relative to χ is the spectral entropy $h_\rho = h_\chi$ of ρ, and the entropy of ρ relative to

$$\varphi = \frac{\chi + \chi \circ \mathrm{i}}{2}$$

is the Hilbert entropy $\mathsf{H}_\rho = h_\varphi$ of ρ. We will prove the Corollary for the spectral entropy, the other being completely analogous.

Theorem C asserts that

$$\alpha h_\rho \leq \frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})} \tag{17}$$

for any simple root $\theta \in \Pi$ of $\mathfrak{a}$ and where $\mathsf{bar}_{\mathfrak{a}^+}$ is the barycenter of the Weyl chamber $\mathfrak{a}^+$. Theorem D implies that equality in (17) can only hold if G is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

Guichard's Theorem gives a finite list of possible groups G, i.e. of possible Zariski closures of $\rho(\pi_1\Sigma)$. We will finish by an explicit computation showing that in all possible cases one has

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})} = \frac{2}{d-1}.$$

The author would like to thank Olivier Guichard for discussions concerning his work.

Theorem 14.1 (Guichard [13]). *Let $\rho : \pi_1\Sigma \to \mathrm{SL}(d,\mathbb{R})$ be the lift of a representation in the Hitchin component, then the Zariski closure $\overline{\rho^Z}$ is either conjugate to $\tau_d(\mathrm{SL}(2,\mathbb{R}))$, $\mathrm{SL}(d,\mathbb{R})$ or conjugate to one of the following groups:*

- $\mathrm{Sp}(2n,\mathbb{R})$ *if* $d = 2n$,
- $\mathrm{SO}(n, n+1)$ *if* $d = 2n+1$,
- G_2 *or* $\mathrm{SO}(3,4)$ *if* $d = 7$.

For $i \in \{1,\dots,k\}$ we will denote by $\varepsilon_i : \mathbb{R}^k \to \mathbb{R}$ the function

$$\varepsilon_i(a_1,\dots,a_k) = a_i.$$

We refer the reader to Knapp's book [16] for the standard computations of simple roots and highest weights that follow.

The $\tau_d(\mathrm{SL}(2,\mathbb{R}))$ **and** $\mathrm{SL}(d,\mathbb{R})$ **cases.** Assume first that $\rho(\pi_1\Sigma)$ is Fuchsian, i.e. it is Zariski dense in $\tau_d(\mathrm{SL}(2,\mathbb{R}))$. A Cartan subspace of $\mathfrak{sl}(2,\mathbb{R})$ is $\mathfrak{a} = \{(a,-a) : a \in \mathbb{R}\}$ the Weyl chamber is $\mathfrak{a}^+ = \{(a,-a) : a \geq 0\}$ with simple root $\Pi = \{2\varepsilon_1\}$. The highest weight of the representation τ_d is $\chi(a,-a) = (d-1)a$. Hence

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})} = \frac{2a}{(d-1)a} = \frac{2}{d-1}.$$

Suppose now that $\rho(\pi_1\Sigma)$ is Zariski dense in $\mathrm{SL}(d,\mathbb{R})$. The Cartan subspace of $\mathfrak{sl}(d,\mathbb{R})$ is $\mathfrak{a} = \{(a_1,\dots,a_d) \in \mathbb{R}^d : a_1 + \cdots + a_d = 0\}$ and

$$\mathfrak{a}^+ = \{(a_1,\dots,a_d) \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\},$$

the simple roots are

$$\Pi = \{\theta_i(a_1,\dots,a_d) = a_i - a_{i+1} : i \in \{1,\dots,d-1\}\}$$

and the barycenter is

$$\mathsf{bar}_{\mathfrak{a}^+} = \{((d-1)t, (d-3)t, \dots, (3-d)t, (1-d)t) : t \geq 0\}.$$

Hence for any $\theta \in \Pi$ one has

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})} = \frac{2t}{(d-1)t} = \frac{2}{d-1}.$$

The $\mathrm{Sp}(2n,\mathbb{R})$ **case.** Assume $d = 2n$ and that the Zariski closure of $\rho(\pi_1\Sigma)$ is $\mathrm{Sp}(2n,\mathbb{R})$. Standard computations show that $\mathfrak{a} = \mathbb{R}^n$, and a Weyl chamber is

$$\mathfrak{a}^+ = \{(a_1,\dots,a_n) : a_i \geq a_{i+1}\ i = 1,\dots,n-1 \text{ and } a_n \geq 0\}.$$

The set of simple roots associated to this Weyl chamber is

$$\Pi = \{\varepsilon_i - \varepsilon_{i+1} : i = 1,\dots,n-1\} \cup \{2\varepsilon_n\}.$$

The barycenter of the Weyl chamber is hence

$$\mathsf{bar}_{\mathfrak{a}^+} = \{((2n-1)t, (2n-3)t, \dots, 3t, t) : t \geq 0\}.$$

The highest weight of the representation $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(d,\mathbb{R})$ is $\chi(a_1,\dots,a_n) = a_1$. Finally, for any $\theta \in \Pi$ one has

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})} = \frac{2t}{(2n-1)t} = \frac{2}{d-1}.$$

The $\mathrm{SO}(n,n+1)$ **case.** Suppose now that $d=2n+1$ and that the Zariski closure of $\rho(\pi_1\Sigma)$ is $\mathrm{SO}(n,n+1)$. Standard computations show that $\mathfrak{a}=\mathbb{R}^n$, and a Weyl chamber is

$$\mathfrak{a}^+=\{(a_1,\ldots,a_n):a_i\geq a_{i+1}\ i=1,\ldots,n-1\text{ and }a_n\geq 0\}.$$

The set of simple roots associated to this Weyl chamber is

$$\Pi=\{\varepsilon_i-\varepsilon_{i+1}:i=1,\ldots,n-1\}\cup\{\varepsilon_n\}.$$

The barycenter of the Weyl chamber is hence

$$\mathsf{bar}_{\mathfrak{a}^+}=\{(nt,(n-1)t,\ldots,2t,t):t\geq 0\}.$$

The highest weight of the representation $\mathrm{SO}(n,n+1)\subset\mathrm{SL}(d,\mathbb{R})$ is $\chi(a_1,\ldots,a_n)=a_1$. Finally, for any $\theta\in\Pi$ one has

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})}=\frac{t}{nt}=\frac{1}{n}=\frac{2}{d-1}.$$

The G_2 **case.** The remaining case is $d=7$ and the Zariski closure of $\rho(\pi_1\Sigma)$ being the exceptional simple Lie group G_2 . We refer the reader to Knapp's book [16, page 692] for the following computations. In this case we have

$$\mathfrak{a}=\{(a_1,a_2,a_3)\in\mathbb{R}^3:a_1+a_2+a_3=0\},$$

a Weyl chamber is

$$\mathfrak{a}^+=\{(a_1,a_2,a_3):a_1\geq a_2\text{ and }-2a_1+a_2+a_3\geq 0\}.$$

The set of simple roots is

$$\Pi=\{\varepsilon_1-\varepsilon_2,-2\varepsilon_1+\varepsilon_2+\varepsilon_3\},$$

and the barycenter of the Weyl chamber is hence

$$\mathsf{bar}_{\mathfrak{a}^+}=\{(-t,-4t,5t):t\geq 0\}.$$

The highest weight associated to the representation $\mathrm{G}_2\to\mathrm{SL}(7,\mathbb{R})$ is

$$\chi=\omega_1=2(\varepsilon_1-\varepsilon_2)-2\varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon_3-\varepsilon_2.$$

Finally, for any $\theta\in\Pi$ one has

$$\frac{\theta(\mathsf{bar}_{\mathfrak{a}^+})}{\chi(\mathsf{bar}_{\mathfrak{a}^+})}=\frac{3t}{5t+4t}=\frac{1}{3}=\frac{2}{d-1}.$$

This finishes the proof.

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